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Dynamical collapse of trajectories

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Abstract – Friction induces unexpected dynamical behaviour. In the paradigmatic pendulum and double-well systems with friction, modelled with differential inclusions, distinct trajectories can collapse onto a single point. Transversal homoclinic orbits display collapse and generate chaotic saddles with forward dynamics that is qualitatively different from the backward dynamics. The space of initial conditions converging to the chaotic saddle is fractal, but the set of points diverging from it is not: friction destroys the complexity of the forward dynamics by generating a unique horseshoe-like topology.

Friction remains one of the poorly understood "f" problems in physics, fracture and fatigue being the others. Throughout history, friction has been dealt with in different ways. The tire industry aims at maximising the friction between the car and the road. Galileo, in his famous inclined-plane experiment to measure the acceleration of gravity experimentally, made every effort to minimise friction. Though friction is ever present in many natural and man-made systems, it induces complex dynamical behaviour which is not yet fully understood. Friction forces are notoriously hard to model, since they are influenced by surface imperfections, wear and debris, crack formation, creep, local stress distribution, and material properties [1–6]. However, qualitative dynamical behaviour of physical oscillators with friction, such as stick-slip behaviour, can be represented by empirical models where the friction force only depends on the slip velocity, and a dry friction (or Amontons-Coulomb) element is used to include a stick phase. Using such models, we show that friction induces novel dynamical behaviour in physical oscillators, and in particular, creates a horseshoe-like object with unexpected dynamics.

Dry friction is described by a force F, which, for non-zero velocities, is constant in magnitude, and its direction is opposite to the velocity. For zero velocities, F can take a range of values. This singular behaviour has very important consequences for the dynamics of physical oscillators. Because F has constant value in magnitude for any non-zero velocity, trajectories of systems with friction can come to rest in finite time, without the exponential approach one is used to in smooth systems. This finite-time convergence causes the violation of the uniqueness of trajectories: different initial conditions can end up in the same point *in finite time*. Equilibria are typically not isolated: an equilibrium point is turned into a one-dimensional manifold of degenerate equilibria by the addition of friction [7].

To visualise this in a classical physical oscillator, consider a pendulum with constant dry friction torque of magnitude T_f acting in the joint, and moment of inertia I. Near the bottom position and for non-zero velocities, the friction torque is dominant and acts in opposite direction of the motion, *i.e.* $I\ddot{\theta} \approx -T_f \,\mathrm{Sign}(\dot{\theta})$. Hence, if the initial velocity is small, then the acceleration is approximately constant and the pendulum comes to rest in finite time. An interval of equilibria exists near the top and bottom position of the pendulum, and all points in these sets attract an infinite set of trajectories in finite time.

In more complex systems displaying chaos, such as the paradigmatic case of the pendulum with forcing, we expect these singular properties of friction to have deep consequences for the global dynamics and the chaotic saddle in

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transient systems [8,9]. In particular, the collapse of trajectories suggests that the future and past dynamics are qualitatively different: only in forward time, trajectories can be continued uniquely from their initial conditions. We will show that this induces a complete asymmetry in the geometry of the stable and the unstable sets of the chaotic saddle. The stable set, corresponding to the set of initial conditions whose orbits converge to the chaotic saddle, is fractal, whereas the unstable set is a regular, non-fractal set. This feature has no counterpart in smooth systems, where the stable and unstable sets are either both fractal (with the same fractal dimensions) or both non-fractal [9]. We will explain how this comes about by adapting the Smale horseshoe [8] to the case of frictional oscillators. The broad impact of our findings is shown in simulations of a paradigmatic system in physics, the perturbed pendulum.

In this letter, we show that in physical systems with friction, homoclinic orbits *collapse* in finite time onto the equilibrium set, creating a homoclinic tangle with a surprising geometry. To show this phenomenon, we consider planar autonomous oscillators with a homoclinic orbit from an equilibrium set and add a small periodic perturbation. As expected from standard dynamical system theory, the separatrix breaks up, creating infinitely many crossings between the set of points converging to the equilibrium set and the set of points emanating from its neighbourhood. We show that the created homoclinic tangle has entirely novel properties compared to homoclinic sets in conventional systems, due to the collapse of trajectories. A return map shows that this new homoclinic tangle generates a chaotic saddle whose forward dynamics is qualitatively different from the dynamics in backward time.

The singular behaviour displayed by frictional oscillators cannot be modelled with ordinary differential equations. Instead, *differential inclusions* are used, where time derivatives of the state are elements of a set-valued right-hand side [10,11]. Solutions of differential inclusions (as defined by Filippov [12,13]) depend continuously on initial conditions. However, differential inclusions are essentially distinct from differential equations in two ways. First, the collapse of trajectories at the discontinuity surface implies that not all solutions are defined uniquely in backward time. Second, differential inclusions exhibit equilibrium sets, *i.e.*, intervals of non-isolated equilibria, *e.g.* a pendulum with friction has an interval of equilibrium points near its top or bottom position. Previous results show that physical systems with friction can be modelled using differential inclusions (see, *e.g.*, [14,15]). Although stability and bifurcations of isolated equilibria or limit cycles in these systems have been studied extensively [16,17], and homoclinic orbits from isolated equilibria are studied, *e.g.*, in [18] for a planar autonomous system, the global phenomenon presented in this letter is still unexplored.

Our results are generic for planar systems with friction that have unstable equilibrium sets, such that homoclinic or heteroclinic orbits can occur. We illustrate them with

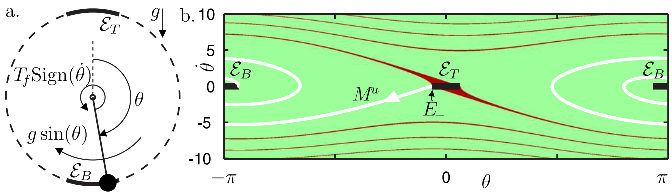

Fig. 1: (Color online) (a) Pictorial of the pendulum. (b) Phase space of system (1) for $(\gamma, \delta) = 0$, with equilibrium sets (black lines) and their basins of attraction, depicted in colors.

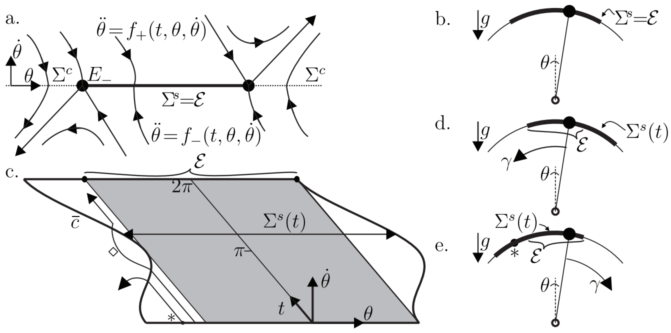

Fig. 2: (a) Local phase portrait of (1) with sets Σ^s and Σ^c for $\gamma = 0$. (b) Pictorial of the upright pendulum. (c) Stick set $\Sigma^s(t)$ and (d), (e) pictorials for $t = 0$ and $t = \pi$ with perturbation $\gamma \neq 0$.

a periodically perturbed pendulum experiencing friction, see fig. 1(a), modelled by the differential inclusion

$$\ddot\theta \in g\sin(\theta) - \delta\dot\theta - T_f\,\text{Sign}(\dot\theta) + \gamma\cos(t), \tag{1}$$

with $(T_f, g) = (2, 10)$, parameters δ, γ and $\text{Sign}(\dot\theta) = \dot\theta/|\dot\theta|$ for $\dot\theta \neq 0$, and $\text{Sign}(0) = [-1, 1]$. Figure 1(b) shows the equilibrium sets $\mathcal{E}_T$ and $\mathcal{E}_B$ of (1) and their basins of attraction without perturbations, $(\gamma = 0)$. Points near the endpoints of an equilibrium set display complex dynamics, while trajectories near other, interior, points of this set behave in a simple fashion, cf. [19]: locally, friction dominates the vector field, such that these trajectories collapse onto the equilibrium set in a finite time. Hence, all equilibrium sets in frictional systems, which are line intervals, are attractive and have a two-dimensional basin of attraction, cf. fig. 1.

Trajectories $\theta(t)$ of the differential inclusion (1) follow smooth vector fields when $\dot\theta < 0$ or $\dot\theta > 0$; let such trajectories be given by $\ddot\theta = f_-(t, \theta, \dot\theta)$ or $\ddot\theta = f_+(t, \theta, \dot\theta)$, respectively, see fig. 2(a). Trajectories stick to the surface where $\dot\theta = 0$, denoted Σ, when the vector fields f_- and f_+ point towards Σ, otherwise, trajectories instantly cross Σ. We denote with $\Sigma^s(t)$ and $\Sigma^c(t)$ the points where trajectories can stick to Σ, or cross through Σ, respectively.

The origin of system (1) corresponds to the top position of the pendulum, see fig. 2(b)–(e). Trajectories can leave the neighbourhood of Σ^s only along the endpoints of Σ^s, other nearby trajectories collapse onto the stick set Σ^s. In the autonomous case $(\gamma = 0)$, all points on Σ^s are equilibria, such that Σ^s coincides with the equilibrium set, denoted $\mathcal{E}$. However, if we apply a torque $\gamma\cos(t)$ to the pendulum, cf. fig. 2(d), (e), then the stick set $\Sigma^s(t)$, shown in panel (c), changes periodically in time. Trajectories near the end of this set stick temporarily, and are released when the perturbation $\gamma\cos(t)$ overcomes the frictional torque, see, *e.g.*, the trajectory from $*$ in fig. 2(c). The equilibrium set $\mathcal{E}$, depicted in gray, consists of the points which are always in $\Sigma^s(t)$.

To study the physical behaviour near a particular endpoint of the equilibrium set, we introduce the stable (or unstable) set, that consists of all initial conditions whose trajectories converge to the endpoint for $t\to\infty$ (or $t\to-\infty$, respectively). Focussing on the left endpoint E_- of the equilibrium set, see fig. 1(b), we observe that f_- is zero at this point when $\gamma=0$, cf. [19]. Hence, trajectories of (1) can leave the neighbourhood of the endpoint exponentially and emanate along the unstable set towards negative velocities, which is denoted M^u.

However, a small periodic perturbation introduces additional complex behaviour: near the endpoint of the equilibrium set, the pendulum can move only for a short time interval. Subsequently, the perturbation will have changed such that the pendulum sticks again, as shown in fig. 2(c) for the trajectory denoted with $\diamond$. Repeating this motion over several periods, eventually, the pendulum leaves the neighbourhood of $\mathcal{E}$. Locally, the unstable set M^u of the endpoint consists of such trajectories.

Both in the autonomous case and in the perturbed case, the stable set of an endpoint, which we denote by M^s, contains trajectories converging to the endpoint in finite time. Since locally, this motion is fast, it is not affected qualitatively by small perturbations. The stable set of an endpoint defines the basin of attraction of the complete equilibrium set, see fig. 1[1]. Comparing the trajectories on the stable and unstable set, we conclude that the discontinuous term in the differential inclusion, which models the physical behaviour of friction, induces different behaviour of the forward and backward dynamics near the equilibrium set.

We argue that a homoclinic or a heteroclinic orbit is generically created from an unstable equilibrium set in physical systems with friction. Such an equilibrium set has an unstable set M^u, such that, typically, sufficiently large perturbations induce a transversal intersection of M^u with the stable set M^s. This intersection creates a homoclinic or heteroclinic tangle, whose unexpected geometry we will explain by studying the break-up of a separatrix by a small periodic perturbation. For the pendulum, typically, a heteroclinic orbit connects the right and left endpoint of the "top" equilibrium set. For ease of exposition, we first study a simpler homoclinic case, which occurs in a perturbed double-well oscillator experiencing friction,

[1]Similar basin boundaries can be observed in fig. 7.7 of [20] for a different system.

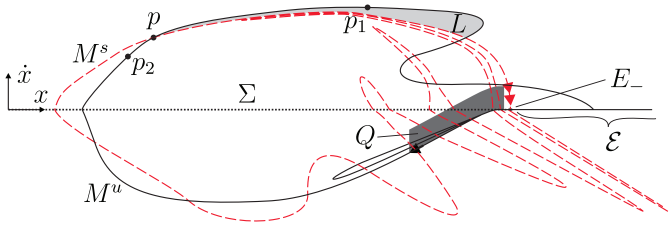

Fig. 3: (Color online) Pictorial illustration of the stable set M^s (dashed) and unstable set M^u (solid) of the endpoint E_-.

modelled with

$$\ddot{x}\in x-x^3-\delta\dot{x}-\gamma\cos(t)-F\,\mathrm{Sign}(\dot{x}), \tag{2}$$

with $F=0.1$ and parameters δ, γ, and introduce $f_+, f_-, \Sigma, \Sigma^s, \Sigma^c$ analogously to the pendulum case. For this system, with $\gamma=0$, a homoclinic orbit occurs when $\delta=\delta^*\approx-0.208$. In this autonomous case, the stable and unstable set coincide to form a homoclinic orbit that is a separatrix. A small perturbation is expected to break up this separatrix, creating a homoclinic tangle of the stable and unstable set, as shown in fig. 3.

The finite-time convergence of trajectories on the stable set dramatically affects the shape of the homoclinic tangle. Analogously to the analysis of homoclinic tangles in periodic differential equations (see, *e.g.*, [21]), consider a stroboscopic map of (2), whose period, denoted T, coincides with the period of the perturbation. Any crossing between the stable and unstable set, *e.g.* point p in fig. 3, corresponds to a homoclinic trajectory that collapses onto the endpoint in finite time. The stroboscopic representation of this trajectory yields a collection of points, that are both contained in M^s and M^u. Due to the finite time convergence, the curve along M^s between the intersection p and the endpoint E_- is crossed only a finite number of times by the unstable set. Trajectories from initial conditions on the unstable set and outside the stable set M^s (*e.g.* point p_1 in fig. 3) remain outside M^s and collapse onto the equilibrium set. Initial conditions on the unstable set and inside the stable set (*e.g.* point p_2 in fig. 3) have trajectories that collapse onto $\Sigma^s(t)$ when they arrive at Σ close enough to the equilibrium set. Further away, such trajectories cross Σ instantly. The forward dynamics of the differential inclusion collapses parts of the "tongues" of the homoclinic tangle (domain L in fig. 3) onto sticking trajectories, or onto the equilibrium set. Since trajectories on the unstable set M^u emanate from the endpoint without a finite-time property, infinitely many crossings occur between the stable and unstable sets. Hence, the homoclinic tangle looks qualitatively as shown in fig. 3.

We show the appearance of a new type of chaotic saddle by studying trajectories from a closed domain Q near the endpoint of the equilibrium set, where Q is bounded on one side by M^u, cf. fig. 3. Trajectories with initial conditions in Q tend to the unstable set M^u, such that the homoclinic tangle implies that some trajectories from Q return to the

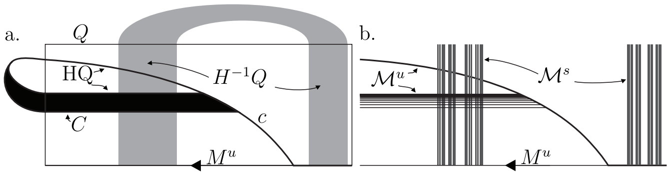

Fig. 4: (a) Pictorial of map H on domain Q. (b) Chaotic saddle with stable set $\mathcal{M}^s$ and unstable set $\mathcal{M}^u$.

same set after a time kT, where an integer k is chosen. Let H denote the map from initial conditions q_0 in Q to the state of the trajectory from q_0 after time kT. To describe the nature of the chaotic saddle, we restrict our attention to trajectories that return to Q during each iteration of H. This is the reformulation of Smale's horseshoe map, see [8], taking the discontinuity and dynamical collapse of our case into account.

The image HQ is described as follows. During the time interval kT, some trajectories from Q experience dynamical collapse: they stick to Σ^s and collide with other trajectories at this surface. At first, we focus on these trajectories. Dynamical collapse occurs when the trajectory arrives in the stick set $\Sigma^s(t)$. Trajectories from points $x \in \Sigma^s(t)$ outside the equilibrium set will slide at time $\tau_-(x)$, where $\tau_-(x)$ is the continuous function that gives the time when $f_-(t,x,0)$ changes sign. In an extended phase space consisting of $(x,\dot{x},t)$, these trajectories are released from stick at the curve $(x,0,\tau_-(x))$ (denoted $\bar{c}$ in fig. 2(c) for the pendulum). Solutions depend continuously on initial conditions, such that the stroboscopic section H maps trajectories from this curve onto a curve c. Hence, the curve $c \subset HQ$ contains all trajectories that experienced collapse in the last perturbation period. Close to the equilibrium set, the curve c coincides with the unstable set M^u of the endpoint, representing trajectories that stick and slide repetitively, like trajectory $\diamond$ in fig. 2(c). Further away, c and M^u separate, since not all trajectories in M^u will have collapsed during the last period. Hence, trajectories from Q that arrive at Σ near E_- experience dynamical collapse and are represented at the Poincaré section by the curve c, shown schematically in fig. 4(a).

Since trajectories from Q tend to the unstable set, which folds back onto itself in the homoclinic tangle, the trajectories that return to Q after time kT emanate from two connected domains in Q, one arriving slightly after the other at the surface Σ. Sufficiently far away from E_-, some trajectories from one of these sets cross Σ through $\Sigma^c(t)$ and remain unique, whereas the other domain collapses completely on $\Sigma^s(t)$. Taking a Poincaré section, the first domain is mapped on a 2-dimensional set C, whereas the latter is mapped to the curve c, cf. fig. 4(a). Hence, $HQ \cap Q$ consists of a line c and a stripe C. The intersection of the forward iterates of this map, *i.e.* $\mathcal{M}^u := \bigcap_{i=0}^{\infty} H^i Q$, shows a set of lines with an accumulation point, and not a fractal structure, see fig. 4(b).

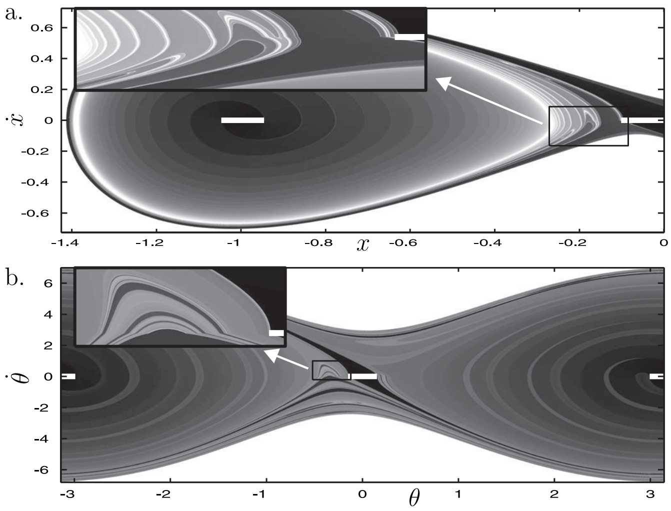

Fig. 5: Residence time before trajectories from a grid of initial conditions arrive at an equilibrium set. Larger residence times are depicted in light gray, equilibrium sets in white. Panel (a) for (2), with $(\gamma,\delta) = (0.002, \delta^*)$, (b) for the pendulum (1) with $(\gamma,\delta) = (0.5, -0.4)$.

The set of initial conditions that return to Q after time kT is given by the preimage $H^{-1}Q$, where $H^{-1}q := \{q_0 | q = H(q_0)\}$. Since the structure of $H^{-1}Q$ is governed by the stable set, analoguous to the Smale horseshoe, $H^{-1}Q \cap Q$ contains two vertical stripes, see fig. 4(a). One iteration further, both stripes are divided in two, such that $\mathcal{M}^s := \bigcap_{i=-\infty}^{0} H^i Q$ is a Cantor set, see fig. 4(b). The backward dynamics of H behaves in a complex manner on a fractal geometry, whereas friction collapses the forward dynamics of H onto a non-fractal structure.

The intersection $\mathcal{X} := \mathcal{M}^s \cap \mathcal{M}^u$ is the Cartesian product of a countable set of points and a Cantor set, and contains an infinite number of points. Since, in addition, any trajectory in $\mathcal{X}$ has a stable and an unstable direction along $\mathcal{M}^s$ and $\mathcal{M}^u$, we refer to the set $\mathcal{X}$ as a chaotic saddle. Note, that in the usual Smale horseshoe, the above intersection is the Cartesian product of two Cantor sets.

Trajectories close to $\mathcal{M}^s$ approach the chaotic saddle $\mathcal{X}$ and spend a long transient time near this set. During this time interval, the chaotic saddle determines their behaviour. Subsequently, they leave the chaotic saddle along the set $\mathcal{M}^u$ which has a simple, non-fractal structure: dynamical collapse destroys the fractal nature of the forward dynamics. Despite this simplified forward dynamics of the physical system, the backward dynamics still behaves in a complex manner. For example, the time spent in the neighbourhood of a chaotic saddle is highly sensitive to initial conditions, cf. fig. 5(a).

Heteroclinic orbits, that exist, *e.g.*, in the pendulum (1), will also generate a chaotic saddle that experiences collapse. Before a trajectory near this saddle returns to a domain close to its initial conditions, it will pass the neighbourhood of two equilibrium sets, and can experience collapse in both of these regions. Although the return map H is tailored to the homoclinic case with only one such domain, near the heteroclinic tangle, collapse also occurs and will induce the asymmetry between the forward and backward dynamics. The backward dynamics near the chaotic saddle remains complex, as illustrated in fig. 5(b), where the time is shown before trajectories of the pendulum converge to one of the equilibrium sets.

In conclusion, we showed that friction in physical systems can induce a homoclinic or heteroclinic tangle with a surprising character. This tangle creates a novel type of chaotic saddle, which induces transient chaos for nearby trajectories. In physical oscillators, dynamical collapse of trajectories generates a qualitative difference between forward and backward dynamics of the saddle: the stable set is fractal, whereas the unstable set is not.

In [22], chaotic saddles in fluidic flows are shown to generate effective mixing, since the unstable manifold is fractal. Our results show, however, that dry friction may cause the final state of trajectories to have a simple geometry, such that chaotic saddles do not yield effective mixing. Apart from physical oscillators with friction studied in this letter, collapse of trajectories is expected in electrical systems, control systems and biological models exhibiting discontinuities, cf. [23–25]. Hence, these systems will also show the asymmetry between forward and backward dynamics.

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