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Bifurcations of equilibrium sets in mechanical systems with dry friction

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ABSTRACT

The presence of dry friction in mechanical systems induces the existence of an equilibrium set, consisting of infinitely many equilibrium points. The local dynamics of the trajectories near an equilibrium set is investigated for systems with one frictional interface. In this case, the equilibrium set will be an interval of a curve in phase space. It is shown in this paper that local bifurcations of equilibrium sets occur near the endpoints of this curve. Based on this result, sufficient conditions for structural stability of equilibrium sets in planar systems are given, and two new bifurcations are identified. The results are illustrated by application to a controlled mechanical system with friction.

1. Introduction

Many mechanical systems experience stick due to dry friction, such that trajectories converge to an equilibrium set, that consists of a continuum of equilibrium points, rather than to an isolated equilibrium point. Dry friction appears at virtually all physical interfaces that are in contact. The presence of equilibrium sets in engineering systems compromises position accuracy in motion control systems, such as robot positioning control; see e.g. [1–3]. Dry friction may also cause other effects that deteriorate performance of motion control systems, such as the occurrence of periodic orbits; cf. [4,3]. In this paper sufficient conditions for structural stability of equilibrium sets are given, and bifurcations of equilibrium sets are studied.

The dry friction force is modelled using a set-valued friction law that depends on the slip velocity, such that the friction law is set-valued only at zero slip velocity. Using such a friction law, the systems are described using a differential inclusion. Such friction laws can accurately describe the existence of an equilibrium set; see e.g. [5–7]. This class of friction models contains the models of Coulomb and Stribeck. Dynamic friction models as discussed in [8] are not considered, since these models increase the dimension of the phase space, such that the study of bifurcations becomes more involved. In the present paper the dynamics of mechanical systems are studied where dry friction is present in one interface.

The equilibrium sets of systems with dry friction may be stable or unstable in the sense of Lyapunov. In addition, equilibrium sets may attract all nearby trajectories in finite time; cf. [6]. A natural question is to ask how changes in system parameters may influence these properties. To answer this question, structural stability and bifurcations of equilibrium sets are studied.

For this purpose, the local phase portrait near an equilibrium set is studied and possible bifurcations are identified. Under a non-degeneracy condition, the local dynamics is shown to be structurally stable near the equilibrium set, except for two specific points, namely the endpoints of the equilibrium set. Hence, analysis of the trajectories near these points yields a categorisation of the bifurcations that are possible. In this manner, particularly, bifurcations of equilibrium sets of planar systems are studied in this paper.

Although quite some results exist on the asymptotic stability and attractivity of equilibrium sets of mechanical systems with dry friction, see [9–12,6,13], few results exist that describe bifurcations of equilibrium sets, see e.g. [14,15]. In [12,6,11], Leine and van de Wouw derive sufficient conditions for attractivity and asymptotic stability of equilibrium sets using Lyapunov theory and invariance results. In the papers of Adly et al. [10,9], conditions are presented under which trajectories converge to the equilibrium set in finite time. Using Lyapunov functions, the attractive properties of individual points in the equilibrium sets are analysed in [13]. Existing results on bifurcations of systems with dry friction are either obtained using the specific properties of a given model, or they do not consider trajectories in the neighbourhood of an equilibrium set. In [14], a van der Pol system is studied that experiences Coulomb friction. By constructing a Poincaré return map, Yabuno et al. show that the limit cycle, created by a Poincaré–Andronov–Hopf bifurcation for the system without friction, cannot be created near the equilibrium set in the presence of friction. In [15], the appearance or disappearance of an equilibrium set is studied by solving an algebraic inclusion.

Bifurcations of the larger family of differential inclusions, that contains models of mechanical systems with friction, are studied e.g. in [16–25]. Bifurcations of limit cycles of discontinuous

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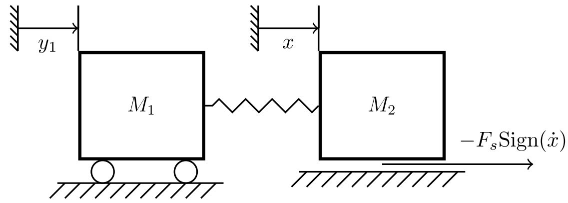


Fig. 1. Mechanical system subject to dry friction.

systems are studied using a return map; see [16,17,22,24]. However, in these systems the friction interface is moving, such that the discontinuity surface does not contain equilibria. Bifurcations of equilibria in two- or three-dimensions are studied in e.g. [18–20,23]. Here, the dynamics is understood by following the trajectories that become tangent to a discontinuity boundary. Guardia et al. present in [25] a generic classification of bifurcations with codimension one and two in planar differential inclusions. However, the special structure of differential inclusions describing mechanical systems with dry friction, which we will analyse in the present paper, is considered to be non-generic by Guardia et al.. In the class of systems studied in the present paper, equilibrium sets occur generically, and persist when physically relevant perturbations are applied. Due to the difference in allowed perturbations, the scenarios observed in the present paper are not considered in [25]. In [17,21], the authors study bifurcations of equilibria in continuous systems, that are not differentiable. Due to the assumptions posed in these papers, all equilibria are isolated points.

In this paper a more general class of mechanical systems with friction is studied. Both the existence of an equilibrium set and the local phase portrait are investigated. Sufficient conditions for structural stability of this phase portrait are given. Such conditions are derived for equilibrium points in smooth systems, among others, by Hirsch, et al.; see [26–28]. Parallel to their approach in smooth systems, we analyse the structural stability of differential inclusions restricted to a neighbourhood of the equilibrium set. At system parameters where the conditions for structural stability are not satisfied, two bifurcations are identified that do not occur in smooth systems.

The outline of this paper is as follows. First, we introduce a model of a mechanical system with friction in Section 2 and present the main result. This theorem states that local bifurcations of equilibrium sets occur in the neighbourhood of two specific points, which are the endpoints of the equilibrium set. In Section 3, classes of systems are identified that are structurally stable. In Section 4, two bifurcations of the equilibrium set of planar systems are presented. In addition, it is shown that no limit cycles can be created by a local scenario similar to the Poincaré–Andronov–Hopf bifurcation. In Section 5, the results of this paper are illustrated with an example of a controlled mechanical system with dry friction. Concluding remarks are given in Section 6. The proofs of the main results are given in Appendices.

2. Modelling and main result

Consider a mechanical system that experiences friction on one interface between two surfaces that move relative to each other in a given direction. Let x denote the displacement in this direction and $\dot{x}$ denote the slip velocity; see Fig. 1 for an example. For an n-dimensional dynamical system this implies that $n-2$ other states y are required besides x and $\dot{x}$. These states contain the other positions and velocities of the mechanical system, and possibly controller and observer states, e.g. in the case of a feedback-controlled motion system. The system given in Fig. 1 can be modelled with the additional states $y = \binom{y_1}{\dot{y}_1}$. In general, using the states x, $\dot{x}$ and y, the dynamics are described by the following differential inclusion; cf. [19]:

$$\begin{aligned} &\ddot{x} - f(x,\dot{x},y) \in -F_s\mathrm{Sign}(\dot{x}), \\ &\dot{y} = g(x,\dot{x},y), \end{aligned} \tag{1}$$

where f and g are sufficiently smooth, $F_s \neq 0$, and $\mathrm{Sign}(\cdot)$ denotes the set-valued sign function

$$\mathrm{Sign}(p) = \begin{cases} \dfrac{p}{|p|}, & p \neq 0, \\ [-1,1], & p = 0. \end{cases}$$

Note that (1) also encompasses systems with other nonlinearities than dry friction, e.g. robotic systems. Introducing the state variables $q = \begin{pmatrix} x & \dot{x} & y^T \end{pmatrix}^T$, the dynamics of (1) can be reformulated as:

$$\dot{q} \in F(q), \tag{2}$$

$$F(q) = \begin{cases} F_1(q), & q \in S_1 := \{q \in \mathbb{R}^n : h(q) < 0\}, \\ F_2(q), & q \in S_2 := \{q \in \mathbb{R}^n : h(q) > 0\}, \\ \mathrm{co}\{F_1(q), F_2(q)\}, & q \in \Sigma := \{q \in \mathbb{R}^n : h(q) = 0\}, \end{cases} \tag{3}$$

where $q \in \mathbb{R}^n$, $\mathrm{co}(a,b)$ denotes the smallest convex hull containing a and b, and F_1 and F_2 and h are given by the smooth functions:

$$F_1(q) = \begin{pmatrix} \dot{x} \\ f(x,\dot{x},y) + F_s \\ g(x,\dot{x},y) \end{pmatrix}, \tag{4}$$

$$F_2(q) = \begin{pmatrix} \dot{x} \\ f(x,\dot{x},y) - F_s \\ g(x,\dot{x},y) \end{pmatrix}, \tag{5}$$

$$h(q) = \dot{x}. \tag{6}$$

In most existing bifurcation results for differential inclusions, see e.g. [18–20,23], parameter changes are considered that induce perturbations of the function F in (2). Hence, in these studies the first component of F is perturbed, which implies that the case where the discontinuity surface coincides with the set where the first element of F is zero is considered non-generic by these authors. This implies that the existence of an equilibrium set in (2) is non-generic. However, parameter changes for the specific system (1) will only yield perturbations of f and g in (2). We show that for the class of systems under study, i.e. mechanical systems with set-valued friction, equilibrium sets will occur, generically.

To study trajectories at the discontinuity surface Σ, the solution concept of Filippov is used; see [19]. Three domains are distinguished on the discontinuity surface. If trajectories on both sides arrive at the boundary, then we have a stable sliding region Σ^s. If one side of the boundary has trajectories towards the boundary, and trajectories on the other side leave the boundary, this domain is called the crossing region Σ^c (or transversal intersection). Otherwise, we have the unstable sliding motion on the domain Σ^u. The mentioned domains are identified as follows:

$$\begin{aligned} &\Sigma := \{q \in \mathbb{R}^n : h(q) = 0\} \\ &\Sigma^s := \{q \in \Sigma : L_{F_1}h > 0 \wedge L_{F_2}h < 0\}, \\ &\Sigma^u := \{q \in \Sigma : L_{F_1}h < 0 \wedge L_{F_2}h > 0\}, \\ &\Sigma^c := \{q \in \Sigma : (L_{F_1}h)(L_{F_2}h) > 0\}, \end{aligned} \tag{7}$$

where $L_{F_i}h$, $i = 1, 2$, denotes the directional derivative of h with respect to F_i, i.e. $L_{F_i}h = \nabla h F_i(q)$.

The vector field $\dot{q} = F^s(q)$ during sliding motion at $q \in \Sigma^u \cup \Sigma^s$ is defined by Filippov as follows. For each q, the vector $F^s(q)$ is the vector on the segment between $F_1(q)$ and $F_2(q)$ that is tangent to Σ at q:

$$\dot{q} = F^s(q) := \frac{L_{F_1}h(q)\,F_2(q) - L_{F_2}h(q)\,F_1(q)}{L_{F_1}h(q) - L_{F_2}h(q)}, \tag{8}$$

$$= \begin{pmatrix} 0 \\ 0 \\ g(x,0,y) \end{pmatrix}. \tag{9}$$

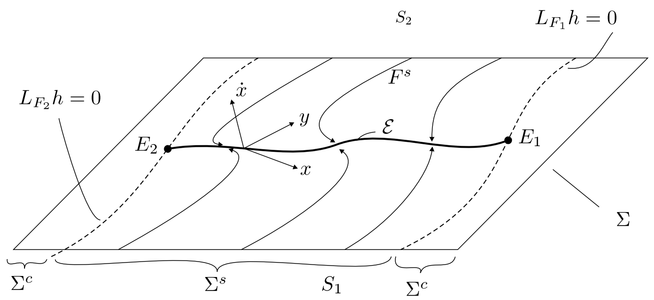


Fig. 2. Sketch of discontinuity surface Σ of (2)–(6) with $n = 3$ and $F_s > 0$, containing an equilibrium set $\mathcal{E}$.

Since $L_{F_1}h = L_{F_2}h + 2F_s$, it follows from (7) that $F_s > 0$ implies that no unstable sliding occurs, and $F_s < 0$ implies that no stable sliding occurs. The resulting phase space is shown schematically in Fig. 2 for the case $n = 3$.

In Appendix A we show that the equilibrium set is a segment of a curve on the discontinuity surface Σ when the following assumption is satisfied.

Assumption 1. The functions f and g are such that $f(0,0,0) = 0$, $g(0,0,0) = 0$ and $\begin{pmatrix} \frac{\partial f(x,0,y)}{\partial x} & \frac{\partial f(x,0,y)}{\partial y} \\ \frac{\partial g(x,0,y)}{\partial x} & \frac{\partial g(x,0,y)}{\partial y} \end{pmatrix}$ is invertible. Furthermore, the map $\binom{f}{g}$ is proper.[1]

For systems satisfying this assumption, the equilibrium set $\mathcal{E}$ of (1) is a one-dimensional curve as shown in Fig. 2. The equilibrium set of a differential inclusion is given by $0 \in F(q)$, which is equivalent with $(q \in \Sigma^s \cup \Sigma^u \wedge L_{F_2}hF_1(q) - L_{F_1}hF_2(q) = 0)$, since $0 \in \begin{pmatrix} q_2 \\ \text{co}\{f(q) - F_s, f(q) + F_s\} \end{pmatrix}$ is equivalent with $q \in \Sigma^s \cap \Sigma^u$ and $g(q) = 0$ is equivalent with $L_{F_2}hF_1(q) - L_{F_1}hF_2(q) = 0$ for $q \in \Sigma^s \cap \Sigma^u$; see (8).

The equilibrium set is divided into interior points and the two endpoints as follows:

$$\begin{aligned} \mathcal{E} &:= \{q \in \Sigma^u \cup \Sigma^s : L_{F_2}hF_1 - L_{F_1}hF_2 = 0\}, \\ I &:= \{q \in \mathcal{E} : F_1 \neq 0 \wedge F_2 \neq 0\}, \\ E_i &:= \{q \in \mathcal{E} : F_i = 0\}, \quad i = 1, 2. \end{aligned} \tag{10}$$

Note, that interior points are called pseudo-equilibria in [17]. The endpoints E_1 and E_2 satisfy $L_{F_1}h = 0$ or $L_{F_2}h = 0$, respectively, hence they are positioned on the boundary of the stable or unstable sliding mode as given by (7).

In the present paper, the trajectories near the equilibrium set will be studied. The influence of perturbations of (1) on the phase portrait of a system is studied. For this purpose, we define the topological equivalence of phase portraits of (1) in Definition 1. We note that this definition is equal to the definition for smooth systems; see e.g. [29–31].

Definition 1 (*[19]*)**.** We say that two dynamical systems in open domains G_1 and G_2, respectively, are *topologically equivalent* if there exist a homeomorphism from G_1 to G_2 which carries, as does its inverse, trajectories into trajectories. This equivalence relation allows for a homeomorphism that does not preserve the parameterisation of the trajectory with time, as required for *topological conjugacy* defined in [29]. Throughout this paper, we assume that f and g smoothly depend on system parameters. When a parameter variation of a dynamical system A yields a system $\tilde{A}$ which is not topologically equivalent to A, then the dynamical system experiences a bifurcation.

[1] A continuous map is proper if the inverse image of any compact set is compact.

With the definitions given above, we can formulate our main result in the following theorem.

Theorem 1. *Assume (1) satisfies Assumption 1. If* $\frac{\partial g}{\partial y}|_p$ *has no eigenvalue* λ *with real*$(\lambda) = 0$ *for any* $p \in \mathcal{E}$*, then the dynamical system (1), in a neighbourhood of the equilibrium set, can only experience bifurcations near the endpoints* E_1 *or* E_2*.*

Proof. The proof is given in Appendix B. □

The theorem is proven using the concept of structural stability, which is introduced in the following section.

3. Structural stability of the system near the equilibrium set

To prove Theorem 1, the influence of perturbations on systems (1) are studied. If perturbations of f and g of (1) cannot yield a dynamical system which is not topologically equivalent to the original system, then the occurrence of bifurcations is excluded. Hence, structural stability of (1) is investigated, which is defined as follows.

Definition 2. A system A given by (1) is *structurally stable for perturbations in f and g* if there exists an $\epsilon > 0$ such that the system $\tilde{A}$ given by (1) with $\tilde{f}$ and $\tilde{g}$ such that

$$\begin{aligned} |f - \tilde{f}| < \epsilon, \quad \left\| \frac{\partial (f - \tilde{f})}{\partial q} \right\| < \epsilon, \\ \|g - \tilde{g}\| < \epsilon, \quad \left\| \frac{\partial (g - \tilde{g})}{\partial q} \right\| < \epsilon, \end{aligned} \tag{11}$$

is topologically equivalent to system A.

Note, that this definition corresponds to C^1-structural stability as defined by Sotomayor [32], and is tailored to dynamical systems described using second-order time derivatives of the state x.

Note that perturbations of (1) in f and g do not cause perturbations of the first component of $F(\cdot)$ in (2), as observed e.g. in [32] or [19, page 226]. One consequence of this fact is that equilibrium sets occur generically in systems (1), although they are non-generic in systems (2). In experiments on mechanical systems with dry friction, such equilibrium sets are found to occur generically; see [8]. For this reason, perturbations of the class (11) are used throughout this paper. System A can be structurally stable for perturbations in f and g, whereas the corresponding system (2) is not structurally stable for general perturbations of F. Small changes of system parameters cause small perturbations of f and g and their derivatives. However, the first equation of (2) will not change under parameter changes. Namely, this equation represents the kinematic relationship between position and velocity of a mechanical system, such that perturbation of this equation does not make sense for the class of physical systems under study. Hence, structurally stability for perturbations in f and g excludes the occurrence of bifurcations near the system parameters studied.

In smooth systems, structural stability of dynamical systems in the neighbourhood of equilibrium points is studied four decades ago by, among others, Hirsch et al.; see [26–28]. It is now well known that, restricted to the neighbourhood of an equilibrium point, smooth dynamical systems are structurally stable when the equilibrium point is hyperbolic. For hyperbolic equilibrium points of smooth systems, the inverse function theorem implies that the equilibrium point is translated over a small distance when the vector field is perturbed, and the perturbed vector field near this equilibrium point is 'close' to the original vector field near the unperturbed equilibrium. Hence, the Hartman–Grobman theorem shows that there exists a topological equivalence for the phase portrait in the neighbourhoods of both equilibria. In this paper, the structural stability of system (1) in the neighbourhood of equilibrium sets is studied analogously.

Small perturbations of system (1) will cause the equilibrium set $\mathcal{E}$ to deform, but the equilibrium set of the perturbed system remains a smooth curve in state space. Hence, there exists a smooth coordinate transformation (analogous to the translation for smooth systems) that transforms the original equilibrium set $\mathcal{E}$ to the equilibrium set of the perturbed system, as shown in Appendix A. Furthermore, the vector field near both equilibrium sets are 'close', as shown in Lemma 7. Using this coordinate transformation, Theorem 1 will be proven in Appendix B.

3.1. Structural stability of planar systems

In this section sufficient conditions are presented for the structural stability of planar systems, restricted to a neighbourhood of equilibrium sets. In the planar case, (1) and (2) reduce to, respectively:

$$\ddot{x} - f(x, \dot{x}) \in -F_s \mathrm{Sign}(\dot{x}), \tag{12}$$

$$\dot{q} \in F(q) = \begin{cases} F_1(q) = \begin{pmatrix} q_2 \\ f(q_1, q_2) + F_s \end{pmatrix}, & h(q) < 0, \\ F_2(q) = \begin{pmatrix} q_2 \\ f(q_1, q_2) - F_s \end{pmatrix}, & h(q) > 0, \end{cases} \tag{13}$$

where $q = \begin{pmatrix} x & \dot{x} \end{pmatrix}^T$ and $h(q) = q_2$. In this case, the Filippov solution $\dot{q} = F_s(q) = 0$ for $q \in \Sigma^s \cup \Sigma^u$, such that the set of interior points of the equilibrium point satisfies $I = \Sigma^s \cup \Sigma^u$.

In this section it is assumed that $F_s > 0$, which corresponds to the practically relevant case where dry friction dissipates energy. The assumption $F_s > 0$ does reduce the number of topologically distinct systems of (12). However, the case $F_s < 0$ yields topologically equivalent systems when time is reversed. The case $F_s > 0$ implies $\Sigma^u = \emptyset$ such that trajectories remain unique in forward time.

According to Theorem 1, structural stability of (12), restricted to a neighbourhood of the equilibrium set, is determined by the trajectories of (12) near the endpoints. Analogously to the Hartman–Grobman theorem, which derives sufficient conditions for structural stability of trajectories near an equilibrium point in smooth systems based on the linearised dynamics near this point, sufficient conditions for structural stability of (12) will be formulated based on the linearisation of F_1 and F_2 near the endpoints of the equilibrium set.

For ease of notation, we define : $A_k := \frac{\partial F_k}{\partial q}|_{q=E_k}$, $k = 1, 2$, which determines the linearised dynamics in S_k near the endpoints of the equilibrium set. In the other domain, i.e. S_{3-k}, it follows from (13) that the vector field is pointing towards the discontinuity surface. To study the structural stability of planar systems (12), we adopt the following assumption.

Assumption 2.

(i) The dry friction force satisfies $F_s > 0$.
(ii) The eigenvalues of A_k, $k = 1, 2$, are distinct and nonzero.

Observe that (i) implies that the equilibrium point $\mathcal{E}$ persists, whereas (ii) concerns the linearised vector field near the endpoints of $\mathcal{E}$. Furthermore, (ii) implies that A_k is invertible. The following theorem presents sufficient conditions for structural stability of (12), restricted to a neighbourhood of the equilibrium set.

Theorem 2. *Consider a system A given by (12) satisfying Assumptions 1 and 2. Restricted to a neighbourhood of the equilibrium set, system A is structurally stable for perturbations in f.*

Proof. The proof is given in Appendix C. □

Table 1
Possible systems (12), categorised by the eigenvalues of the Jacobian matrix near the endpoints, which are locally structurally stable in a neighbourhood of the equilibrium set. Note that the indices of A_1 and A_2 can be changed. Eigenvalues are denoted with − or + when the eigenvalues are real and positive or negative, respectively. Complex eigenvalues are denoted with c.

	Eigenvalues of A_1		Eigenvalues of A_2	
	λ_1	λ_2	λ_1	λ_2
Sink–Sink	−	−	−	−
Sink–Source	−	−	+	+
Sink–Saddle	−	−	−	+
Sink–Focus	−	−	c	c
Source–Source	+	+	+	+
Source–Saddle	+	+	−	+
Source–Focus	+	+	c	c
Saddle–Saddle	−	+	−	+
Saddle–Focus	−	+	c	c
Focus–Focus	c	c	c	c

The proof of this theorem is given in Appendix C. The theorem implies that one can identify 10 different types of systems (12) with a stable sliding mode which, restricted to a neighbourhood of the equilibrium set, are locally structurally stable, as shown in Table 1. Note, that an unstable sliding mode yields analogous types of equilibrium sets.

If f satisfies the symmetry property: $f(x, 0) = -f(-x, 0)$, then only the Source–Source, Sink–Sink, Saddle–Saddle and Focus–Focus types are possible.

4. Bifurcations

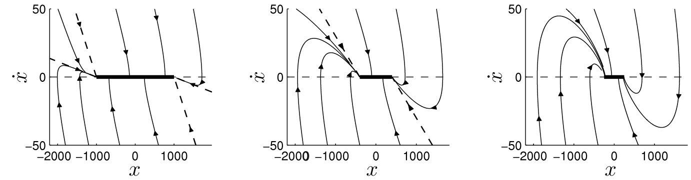
Due to the special structure of (1), general dynamical systems of the form (2) have a richer dynamics than systems (1). For example, all codimension-one bifurcations of (2) as observed in [20] cannot occur in (1). In general, the sliding motion of the system (2) yields a nonzero sliding vector field, whereas the sliding motion of (1) contains a set of equilibria. Therefore, in this section bifurcations of (1) are studied, restricting ourselves to planar systems as given in (12). Such bifurcations occur in systems (2) as well, but will have a higher codimension.

Using Assumption 2, in Section 3 several types of topologically distinct planar systems (12) are identified, which are structurally stable in a neighbourhood of the equilibrium set. Hence, it seems a reasonable step to consider parameter-dependent systems, and study the parameters where the given conditions on the differential inclusion (12) no longer hold. In this manner, two bifurcations of planar systems (1) are presented. At the bifurcations points, Assumption 2(ii) is violated.

4.1. Real or complex eigenvalues

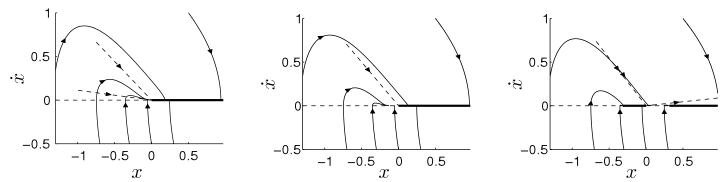
Consider system (12) where eigenvalues of A_1 change from real to complex eigenvalues under a parameter variation. From (12) it follows that $A_1 = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$, hence the eigenvalues of A_1 are distinct when $a_{21} \neq -\frac{1}{4}a_{22}^2$ and the eigenvalues are both nonzero given $a_{21} \neq 0$. Now, let the first part of Assumption 2 be violated, such that the eigenvalues are equal. In that case, we obtain $a_{21} = -\frac{1}{4}a_{22}^2$.

Arbitrarily close to systems with $a_{21} = -\frac{1}{4}a_{22}^2$ there exist topologically distinct systems, since a system where A_1 has complex eigenvalues is topologically distinct from a system where A_1 has real eigenvalues. This follows from the observation that there exists a stable or unstable manifold containing E_1 if and only if A_1 has real eigenvalues. However, there exists no homeomorphism that satisfies the conditions in Definition 1 and maps a trajectory on this manifold to a trajectory of a system where



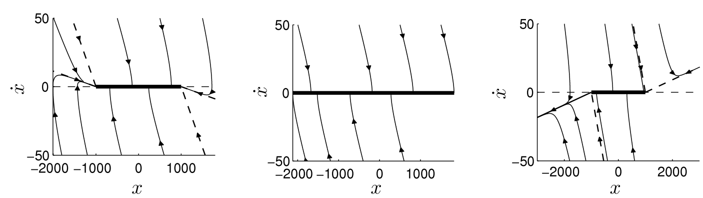
(a) $a_{21} = -0.001$. (b) $a_{21} = -0.0025$. (c) $a_{21} = -0.004$.

Fig. 3. System (14) with $F_s = 1$ and $a_{22} = -0.1$ showing a focus-node bifurcation. The equilibrium set $\mathcal{E}$ is given by a bold line, the real eigenvectors of the stable eigenvalues of A are represented with dashed lines. The real eigenvectors are distinct for $a_{21} > -0.0025$, collide at a_{21} and subsequently become imaginary.



(a) $a_{21} = -0.1$. (b) $a_{21} = 0$. (c) $a_{21} = 0.1$.

Fig. 4. System (15) with $F_s = 1$, $a_{22} = -1$ and varying a_{21}, showing a bifurcation where an eigenvalue becomes zero. A neighbourhood of the origin is depicted, that does not contain the complete equilibrium set. The equilibrium set $\mathcal{E}$ is given by a bold line, the eigenvectors of stable or unstable eigenvalues of $\frac{\partial F_2}{\partial q}|_{q=0}$ are represented with dashed lines.



(a) $a_{21} = -0.001$. (b) $a_{21} = 0$. (c) $a_{21} = 0.001$.

Fig. 5. System (14) with $F_s = 1$, $a_{22} = -0.1$ and varying a_{21}, showing a bifurcation when an eigenvalue becomes zero and the system is linear. The equilibrium set $\mathcal{E}$ is given by a bold line, the stable or unstable manifolds are represented with dashed lines. At $a_{21} = 0$, the equilibrium set coincides with the line $x \in \mathbb{R}, \dot{x} = 0$.

A_1 has complex eigenvalues, since in that case only one trajectory converges towards E_1, which originates from S_2.

This bifurcation is illustrated with the following exemplary system:

$$\ddot{x} - a_{21}x - a_{22}\dot{x} \in -F_s \mathrm{Sign}(\dot{x}), \tag{14}$$

with $F_s = 1$, $a_{22} = -0.1$ and varying a_{21}. In this example the matrices A_1 and A_2 are equal, such that both endpoints undergo a bifurcation at the same time. This system shows a bifurcation when $a_{21} = -0.0025$, as shown in Fig. 3. We refer to this bifurcation as a focus-node bifurcation. According to [9], all trajectories of system (14) will arrive in the equilibrium set $\mathcal{E}$ in finite time if and only if $a_{21} < -\frac{1}{4}a_{22}^2 = -0.0025$. For $a_{21} \geq 0.0025$, the matrices A_1 and A_2 have a real eigenvector corresponding to an eigenvalue λ. The span of this eigenvector contains trajectories that converge exponentially according to $x(t) - E_i = e^{\lambda t}(x(0) - E_i)$, $i = 1, 2$, which consequently does not converge in finite time. Hence, this change of the attractivity properties of the equilibrium set coincides with a bifurcation, defined using topological equivalence as used in this paper.

4.2. Zero eigenvalue

Consider system (12) where an eigenvalue of A_1 becomes zero under parameter variation, where $A_1 = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$. This matrix has an eigenvalue equal to zero when $a_{21} = 0$ becomes zero.

At the point E_1 of the equilibrium set the vector field satisfies $F_1(E_1) = 0$. By definition, the point E_1 is an endpoint of $\mathcal{E}$, such that trajectories cross Σ outside $\mathcal{E}$. This implies that the second component of F_1, denoted $F_1^{\dot{x}}$, evaluated on the curve Σ changes sign at E_1. Since F_1 is smooth and $F_1^{\dot{x}}$ changes sign at E_1, we obtain $\frac{\partial^k F_1^x}{\partial x^k}|_{E_1} \neq 0$ for an odd integer $k \geq 3$, and $\frac{\partial^i F_1^x}{\partial x^i}|_{E_1} = 0$, for $i = 1, \ldots, k-1$. The equilibrium set $\mathcal{E}$ on Σ is given by $F_2^{\dot{x}} < 0$ and $F_1^{\dot{x}} > 0$. A change of a system parameter can create two distinct domains where $F_1^{\dot{x}} > 0$ near E_1, such that two equilibrium sets are created.

This bifurcation is illustrated with the following exemplary system:

$$\ddot{x} - a_{21}x - a_{22}\dot{x} + F_s + x^3 \in -F_s \mathrm{Sign}(\dot{x}), \tag{15}$$

with $F_s = 1$, $a_{22} = -1$ and varying a_{21}. The system is designed such that the origin is always the endpoint of an equilibrium set. The resulting phase portrait is given in Fig. 4, and shows the mentioned bifurcation. For $a_{21} = -0.1$, one compact equilibrium set exists. For $a_{21} = 0$, an eigenvalue of the system becomes zero, and the corresponding eigenvector is parallel to the equilibrium set. Note that this implies that both Assumptions 1 and 2(ii) are violated. For $a_{21} > 0$, the equilibrium set splits in two separated, compact, equilibrium sets; cf. Fig. 4(c).

Another bifurcation occurs when F_1 and F_2 are linear systems. In that case, the equilibrium set grows unbounded when $a_{21} \to 0$, and becomes the complete line satisfying $x \in \mathbb{R}, \dot{x} = 0$. This bifurcation is illustrated in Fig. 5 using system (14) with $a_{22} = -0.1$, $F_s = 1$ and varying a_{21}.

4.3. Closed orbits

In smooth systems the Poincaré–Andronov–Hopf bifurcation can create a small closed orbit near an equilibrium point. A similar scenario cannot occur in planar systems (12) when $F_s \neq 0$, as shown in the following lemma.

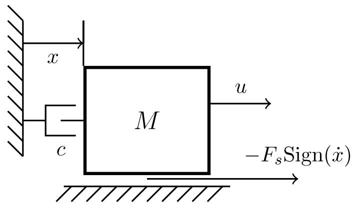


Fig. 6. Example of mechanical system subject to dry friction.

Lemma 3. *Consider system (12) with $F_s \neq 0$ satisfying Assumption 1 which has closed orbit γ with period T_γ. Given a period time T_γ, there exists an $\epsilon > 0$ such that $\mathcal{E} + \mathcal{B}_\epsilon$ does not contain parts of the closed orbit γ.*

Proof. The proof is given in Appendix D. □

This lemma contradicts the appearance of a limit cycle with finite period near the equilibrium set, as appears close to a smooth Poincaré–Andronov–Hopf bifurcation point in smooth systems. Note, that the appearance of limit cycles is not excluded, when changes of system parameters causes the dry friction force F_s to change sign, which is physically unrealistic. When the discontinuous nature of the system is introduced by other effects than dry friction, the appearance of limit cycles can occur in physical systems; see e.g. [33].

Remark 1. A heteroclinic orbit may exist that connects the endpoints E_1 to E_2 through two trajectories, one positioned in the smooth domain S_1 and the other positioned in the opposite smooth domain S_2. Small perturbations of this system can be expected to cause the appearance of limit cycles with an arbitrary large period time.

The result given in Lemma 3 is derived using the specific structure of the vector field near the equilibrium set. Non-local events such as the appearance of homoclinic or heteroclinic orbits are not considered in this paper, and will be subject to further research.

5. Illustrative example

The applicability of Theorem 1 for higher-dimensional systems is illustrated with an observer-based control system, where a single mass is controlled using a velocity observer. The system is given by:

$$\dot{q} = Aq + B(u + f(q)), \tag{16}$$

with $q = \begin{pmatrix} x & \dot{x} \end{pmatrix}^T \in \mathbb{R}^2$, measurement $z = x$, control input u and friction force $f(q) \in -F_s\mathrm{Sign}(q_2)$, as shown schematically in Fig. 6. We assume $M = 1$. The matrix A is given by $A = \begin{pmatrix} 0 & 1 \\ 0 & -c \end{pmatrix}$, with $c > 0$ and $B = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$.

For this system, a linear state feedback controller of PD-type is designed, yielding $u = k_p z + k_d v$, with proportional gain k_p, differential gain k_d, and v an estimate of the velocity $\dot{x}$. This estimate is obtained with the following reduced order observer, that is designed for the linear system without friction; see [34]:

$$\dot{v} = -cv + u. \tag{17}$$

After substitution of $v = y$, the resulting closed-loop system is given by

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dot{y} \end{pmatrix} \in A_c \begin{pmatrix} x \\ \dot{x} \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -F_s\mathrm{Sign}(\dot{x}) \\ 0 \end{pmatrix}, \tag{18}$$

$$A_c = \begin{pmatrix} 0 & 1 & 0 \\ -k_p & -c & -k_d \\ -k_p & 0 & -c - k_d \end{pmatrix}, \tag{19}$$

which is equivalent with (2), where $f(x, \dot{x}, y) = -k_p x - c\dot{x} - k_d y$ and $g = -k_p x - (c + k_d)y$. Assumption 1 implies $ck_p \neq 0$. If this is satisfied, system (19) has the equilibrium set $\{q = \begin{pmatrix} x & \dot{x} & y \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} x & \dot{x} & y \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{k_p} + \frac{k_d}{ck_p}\right)\alpha & 0 & \frac{\alpha}{c} \end{pmatrix}, \alpha \in [-F_s, F_s]\}$.

Since $\frac{\partial g}{\partial y} = -c - k_d$, Theorem 1 shows that when $-c - k_d \neq 0$, no bifurcations occur away from the endpoints E_1 and E_2, given by $\begin{pmatrix} x & \dot{x} & y \end{pmatrix} = \pm \begin{pmatrix} -\left(\frac{1}{k_p} + \frac{k_d}{ck_p}\right)F_s & 0 & \frac{F_s}{c} \end{pmatrix}$.

The structural stability of trajectories near the endpoints of an equilibrium set is studied in the present paper only for planar systems, while the current example is 3-dimensional. However, we will still present a bifurcation of trajectories near the endpoints. Similar to the approach used in Section 4, the linearisation of the vector field near the endpoints is used. Here, matrices A_1 and A_2 coincide with A_c, which has eigenvalues $\lambda_1 = -c$ and $\lambda_{2,3} = -\frac{c+k_d}{2} \pm \frac{1}{2}\sqrt{(c + k_d)^2 - 4k_p}$. The eigenvalues $\lambda_{2,3}$ change from real to complex when $k_d = -c + 2\sqrt{k_p}$. At this point a bifurcation occurs similar to the focus-node bifurcation observed in Section 4.1. When two eigenvalues are complex, for both endpoints E_i, $i = 1, 2$, there exists only one trajectory that converges to the endpoints E_i from domain S_i for $t \to \infty$ or $t \to -\infty$. When eigenvalues $\lambda_{2,3}$ are real, more trajectories exist with this property. Hence, a bifurcation occurs when k_d crosses the value $-c + 2\sqrt{k_p}$. This bifurcation is illustrated in Fig. 7 where the parameters $c = 0.5$, $k_p = 1$, $F_s = 2$ are used. At these parameters, the mentioned bifurcation occurs at $k_d = 1.5$. For the used system parameters, the eigenvalues of A_c have negative real part. In Fig. 7, only trajectories near the endpoint E_2 are shown. Since the system is symmetric, the same bifurcation occurs near the endpoint E_1.

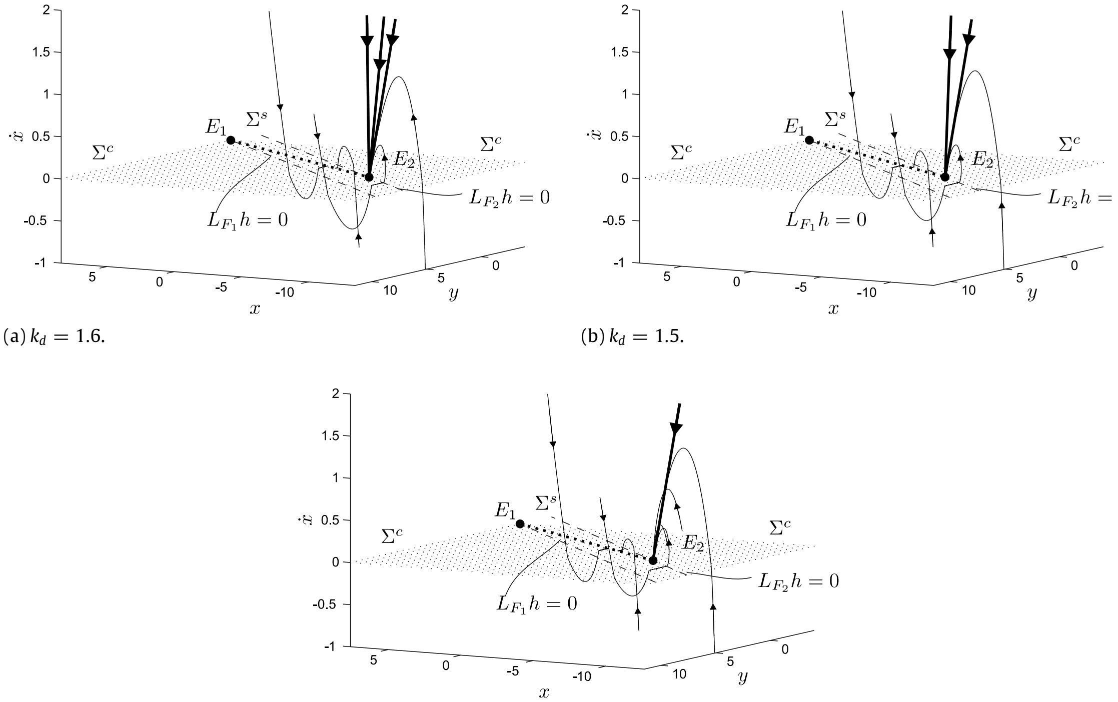
These results suggest that using the linearisation of the dynamics near the endpoints, sufficient conditions for structural stability of trajectories can be constructed for higher-dimensional systems, analogously to the results in Sections 3.1 and 4 for planar systems.

6. Discussion

In this paper, bifurcations and structural stability of a class of nonlinear mechanical systems with dry friction are studied in the neighbourhood of equilibrium sets. It has been shown in Theorem 1 that local bifurcations of equilibrium sets of a class of nonlinear mechanical systems with a single frictional interface can be understood by studying the trajectories of two specific points in phase space, which are the endpoints of the equilibrium set. Hence, local techniques can be applied in a neighbourhood of these points. For differential inclusions given by (1), the linearisation of vector fields is only applicable to the part of the state space where the vector field is described by a smooth function. A careful study of this linearisation has given insight in the topological nature of solutions of the differential inclusion near the equilibrium set. Hence, in the neighbourhood of equilibrium sets the result of Theorem 1 significantly simplifies the further study of structural stability and bifurcations for this class of mechanical systems with friction.

Using this approach, sufficient conditions are derived for structural stability of planar systems given by (1), restricted to a neighbourhood of the equilibrium set. Furthermore, two types of bifurcations of the equilibrium set of this class of systems are identified, which do not occur in smooth systems.

Discontinuous systems have been studied in [35,36] using a smooth approximation of the discontinuity, followed by the use of singular perturbation theory to obtain the dynamics on a slow manifold. If this approach is followed for system (1), then the equilibrium set is represented by an equilibrium point on the slow manifold. Investigating the similarities between these approaches would be an interesting direction for further research.



(a) $k_d = 1.6$. (b) $k_d = 1.5$. (c) $k_d = 1.4$.

Fig. 7. System (19) with $c = 0.5$, $k_p = 1$ and $F_s = 2$, showing a bifurcation near the endpoints at $k_d = 1.5$. The equilibrium set $\mathcal{E}$ is given by a dotted line, and the real eigenvectors of A_2 are represented with thick lines.

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Appendix A. Existence of an equilibrium set

In this section the existence of an equilibrium set is shown for system A given by (1) and a perturbed system $\tilde{A}$. Subsequently, the existence is proven of a smooth coordinate transformation that maps the equilibrium set of A onto the equilibrium set of a perturbed system $\tilde{A}$. The section is concluded with a technical result on the dynamics of A expressed in the new coordinates. This result will show that we may assume that the equilibrium sets of A and $\tilde{A}$ coincide, without influencing the conditions posed in Theorem 1.

In the remainder of this paper, let $\tilde{A}$ be a perturbed system given by (1) with $\tilde{f}$ and $\tilde{g}$ perturbed versions of f and g, respectively. Let the sets $\tilde{\mathcal{E}}$, $\tilde{I}$, $\tilde{S}_1$, $\tilde{S}_2$, $\tilde{\Sigma}$, $\tilde{\Sigma}^c$, $\tilde{\Sigma}^s$, functions $\tilde{F}(\cdot)$ $\tilde{F}_1(\cdot)$, $\tilde{F}_2(\cdot)$, $\tilde{F}_s(\cdot)$ and points $\tilde{E}_1$ and $\tilde{E}_2$ of system $\tilde{A}$ be defined analogous to the sets, functions and points defined for system A.

The following result shows that the equilibrium sets $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are curves in the state space.

Lemma 4. *Consider system A and $\tilde{A}$ given by (1) with $f, g, \tilde{f}, \tilde{g}$ satisfying (11) for $\epsilon > 0$ sufficiently small. Furthermore, let Assumption 1 be satisfied. The equilibrium sets $\mathcal{E}$ and $\tilde{\mathcal{E}}$ of systems A and $\tilde{A}$, respectively, are curves in state space that can be parameterised by smooth functions c and $\tilde{c}$, such that $\mathcal{E} = \{c(\alpha), \alpha \in [-F_s, F_s]\}$, $f(c(\alpha)) = \alpha$ and $g(c(\alpha)) = 0$, resp. $\tilde{\mathcal{E}} = \{\tilde{c}(\alpha), \alpha \in [-F_s, F_s]\}$, $\tilde{f}(\tilde{c}(\alpha)) = \alpha$ and $\tilde{g}(\tilde{c}(\alpha)) = 0$.*

Proof. Define $Z(x, y) = \begin{pmatrix} f(x,0,y) \\ g(x,0,y) \end{pmatrix}$ and observe that $0 \in F(q)$ for all $q \in \mathcal{E}$ implies $Z(x, y) = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$, with $\alpha \in [-F_s, F_s]$. Using Assumption 1, the global inverse function theorem; cf. [37], can be applied, which states that $Z(x, y)$ is a homeomorphism. Application of the Corollary following Lemma 2 of [37] shows that the inverse of the function Z is smooth. Hence, there exist smooth functions $X(\beta)$ and $Y(\beta)$ defined by $Z(X(\beta), Y(\beta)) = \beta$.

Since (11) is satisfied by A and $\tilde{A}$ with $\epsilon > 0$ sufficiently small, we find that for sufficiently small ϵ, the matrix $\begin{pmatrix} \frac{\partial \tilde{f}(x,0,y)}{\partial x} & \frac{\partial \tilde{f}(x,0,y)}{\partial y} \\ \frac{\partial \tilde{g}(x,0,y)}{\partial x} & \frac{\partial \tilde{g}(x,0,y)}{\partial y} \end{pmatrix}$ is invertible. Define $\tilde{Z}(x, y) := \begin{pmatrix} \tilde{f}(x,0,y) \\ \tilde{g}(x,0,y) \end{pmatrix}$. The functions $\tilde{f}$ and $\tilde{g}$ are smooth, such that application of the global inverse function theorem yields that there exists an $\epsilon > 0$ such that (11) implies that there exist smooth functions $\tilde{X}$ and $\tilde{Y}$ such that $\tilde{Z}(\tilde{X}(\beta), \tilde{Y}(\beta)) = \beta$. Now, we define $c_x(\alpha) = X\left(\begin{pmatrix} \alpha \\ 0 \end{pmatrix}\right)$ and $c_y(\alpha) = Y\left(\begin{pmatrix} \alpha \\ 0 \end{pmatrix}\right)$ and $c(\alpha) = \begin{pmatrix} c_x(\alpha) \\ 0 \\ c_y(\alpha) \end{pmatrix}$, such that $Z(c_x(\alpha), c_y(\alpha)) = (\alpha, 0)$. The curve $c(\cdot)$ is a parameterisation of $\mathcal{E}$, such that $\mathcal{E} = \{q \in \mathbb{R}^n : q = c(\alpha), \alpha \in [-F_s, F_s]\}$. Analogously, a parameterisation $\tilde{c}(\cdot)$ of $\tilde{\mathcal{E}}$ can be constructed. □

Using this lemma, a map is constructed that maps the equilibrium set $\mathcal{E}$ of A onto the equilibrium set $\tilde{\mathcal{E}}$ of $\tilde{A}$.

Lemma 5. *Consider system A and $\tilde{A}$ given by (1) with $f, g, \tilde{f}, \tilde{g}$ satisfying (11) for $\epsilon > 0$ sufficiently small. Furthermore, let Assumption 1 be satisfied and let $\frac{\partial g}{\partial y}|_p$ be invertible for all $p \in \mathcal{E}$. There exists a smooth map $H_e = \begin{pmatrix} H_e^x \\ H_e^{\dot{x}} \\ H_e^y \end{pmatrix}$ with smooth inverse in a neighbourhood of $\mathcal{E}$, such that $H_e^x = H_e^x(x)$ and $H_e^{\dot{x}} = \frac{\mathrm{d}H_e^x}{\mathrm{d}x}\dot{x}$, that maps $\mathcal{E}$ onto $\tilde{\mathcal{E}}$. Furthermore, for any $\delta > 0$ there exists an $\epsilon > 0$ such that for any system $\tilde{A}$ for which (11) holds, the resulting map H_e satisfies $\|H_e(q) - q\| < \delta$ and $\|\frac{\partial H_e}{\partial q} - I\| < \delta$, where I denotes the n-dimensional identity matrix.*

Proof. In this proof the map H_e is constructed as follows. Using Lemma 4, it will be shown that the equilibrium sets $\mathcal{E}$ and $\tilde{\mathcal{E}}$ can be parameterised as functions of x and $\tilde{x}$, respectively. Using this parameterisation, a map H_e is constructed that maps the curve $\mathcal{E}$ onto $\tilde{\mathcal{E}}$.

Consider the parameterisations $c(\cdot)$ and $\tilde{c}(\cdot)$ of $\mathcal{E}$ and $\tilde{\mathcal{E}}$, respectively, as given in Lemma 4. In order to construct a map H_e which is defined in a neighbourhood of $\mathcal{E}$, we extend $c(\alpha)$ and $\tilde{c}(\alpha)$ such that they are defined in a neighbourhood of a closed set $\alpha \in [\alpha_1, \alpha_2]$, with $\alpha_1 < -|F_s|$ and $\alpha_2 > |F_s|$, which is possible according to the extension lemma; see [38].

Let $c(\alpha) = \begin{pmatrix} c_x(\alpha) \\ 0 \\ c_y(\alpha) \end{pmatrix}$ and $\tilde{c}(\alpha) = \begin{pmatrix} \tilde{c}_x(\alpha) \\ 0 \\ \tilde{c}_y(\alpha) \end{pmatrix}$. Differentiating the defining expressions $f(c_x(\alpha), 0, c_y(\alpha)) = \alpha$ and $g(c_x(\alpha), 0, c_y(\alpha)) = 0$ with respect to α yields $\frac{\partial f}{\partial x}\frac{\mathrm{d}c_x}{\mathrm{d}\alpha} + \frac{\partial f}{\partial y}\frac{\mathrm{d}c_y}{\mathrm{d}\alpha} = 1$, and $\frac{\partial g}{\partial x}\frac{\mathrm{d}c_x}{\mathrm{d}\alpha} + \frac{\partial g}{\partial y}\frac{\mathrm{d}c_y}{\mathrm{d}\alpha} = 0$, which should be satisfied at the equilibrium set. Since both equations should be satisfied, invertibility of $\frac{\partial g}{\partial y}$ implies $\frac{\mathrm{d}c_x}{\mathrm{d}\alpha} \neq 0$ along the equilibrium curve. Smoothness of g and c imply that one can pick $\alpha_1 < -|F_s|$ and $\alpha_2 > |F_s|$ sufficiently close to $-F_s$ and F_s, such that $\frac{\mathrm{d}c_x}{\mathrm{d}\alpha}|_\alpha \neq 0$ and $\frac{\partial g}{\partial y}|_{c(\alpha)}$ is invertible for all $\alpha \in (\alpha_1, \alpha_2)$. The function $c_x(\alpha)$ is smooth, such that the $\frac{\mathrm{d}c_x}{\mathrm{d}\alpha}|_\alpha \neq 0$ implies that the inverse function theorem can be applied, which yields that there exists a smooth inverse $c_x^{-1}(x)$ defined on the interval $x \in \mathrm{co}\{c_x(\alpha_1), c_x(\alpha_2)\}$. Furthermore, $\frac{\mathrm{d}c_x}{\mathrm{d}\alpha} \neq 0$ implies that c_x is either monotonously increasing or decreasing, such that the x-variables of $\mathcal{E}$ are positioned in $\mathrm{co}\{c_x(-F_s), c_x(F_s)\}$. Now, let $y = \Psi(x) := c_y(c_x^{-1}(x))$ denote the y-coordinate such that the equilibrium set $\mathcal{E}$ can be parameterised as follows:

$$\mathcal{E} = \{q \in \mathbb{R}^n : q = \begin{pmatrix} x & 0 & \Psi^T(x) \end{pmatrix}^T, \quad x \in \mathrm{co}\{c_x(-F_s), c_x(F_s)\}\}. \tag{A.1}$$

We observe that $c_x^{-1}(x)$ is defined on the interval $x \in \mathrm{co}\{c_x(\alpha_1), c_x(\alpha_2)\}$ and $c_y(\alpha)$ is defined on (α_1, α_2). Hence, the map $\Psi(x)$ is defined in a neighbourhood of $\mathrm{co}\{c_x(-F_s), c_x(F_s)\}$.

To construct a function $\tilde{\Psi}(\cdot)$ for the equilibrium set $\tilde{\mathcal{E}}$ analogous to $\Psi(\cdot)$, first we will prove that $\frac{\partial \tilde{g}}{\partial y}|_{\tilde{p}}$ is invertible for $\tilde{p} \in \tilde{\mathcal{E}}$ if ϵ is sufficiently small. Since the functions $f, g, \tilde{f}$, and $\tilde{g}$ are smooth and satisfy Assumption 1, the inverse function theorem implies that for each $\epsilon_1 > 0$ there exists an $\epsilon > 0$ such that (11) implies that for each p satisfying $\begin{pmatrix} p_2 & f(p) & g(p)^T \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \end{pmatrix}$ there exists a $\tilde{p}$, such that $\begin{pmatrix} \tilde{p}_2 & \tilde{f}(\tilde{p}) & \tilde{g}(\tilde{p})^T \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \end{pmatrix}$ and $\|p - \tilde{p}\| < \epsilon_1$. Hence, there exists an $\epsilon > 0$, such that, firstly, $\|\tilde{p} - p\| < \epsilon_1$ and secondly, invertibility of $\frac{\partial \tilde{g}}{\partial y}|_p$ implies invertibility of $\frac{\partial \tilde{g}}{\partial y}|_p$. If ϵ is sufficiently small, this implies that $\frac{\partial \tilde{g}}{\partial y}|_{\tilde{p}}$ is invertible.

Therefore a function $\tilde{\Psi}$ can be constructed for the equilibrium set $\tilde{\mathcal{E}}$ analogously to the function Ψ for equilibrium set $\mathcal{E}$ when $\epsilon > 0$ is sufficiently small, such that the equilibrium set $\tilde{\mathcal{E}}$ can be parameterised as follows:

$$\tilde{\mathcal{E}} = \{q \in \mathbb{R}^n : q = \begin{pmatrix} \tilde{x} & 0 & \tilde{\Psi}^T(\tilde{x}) \end{pmatrix}^T, \quad \tilde{x} \in \mathrm{co}\{\tilde{c}_x(-F_s), \tilde{c}_x(F_s)\}\}. \tag{A.2}$$

Let E_1^x, E_2^x denote the x-component of E_1 and E_2, respectively, such that $E_1^x = c_x(-F_s)$, and $E_2^x = c_x(F_s)$. Hence, E_1^x and E_2^x are the x-coordinates of the endpoints of the equilibrium set $\mathcal{E}$. The values $\tilde{E}_1^x$, and $\tilde{E}_2^x$ are defined analogously. The smooth invertible map H_e is constructed as follows. Let $H_e^x(x) = \tilde{E}_1^x + \frac{(x-E_1^x)(\tilde{E}_2^x - \tilde{E}_1^x)}{E_2^x - E_1^x}$ and $H_e^{\dot{x}}(\dot{x}) = \frac{(\tilde{E}_2^x - \tilde{E}_1^x)}{E_2^x - E_1^x}\dot{x}$. Furthermore, let $H_e^y(x, y) = y + \tilde{\Psi}(H_e^x(x)) - \Psi(x)$.

Since $\Psi(x)$ and $\tilde{\Psi}(\tilde{x})$ are defined in neighbourhoods of $x \in \mathrm{co}\{E_1^x, E_2^x\}$ and $\tilde{x} \in \mathrm{co}\{\tilde{E}_1^x, \tilde{E}_2^x\}$, respectively, we observe that H_e is defined in a neighbourhood of $\mathcal{E}$. The maps Ψ and $\tilde{\Psi}$ are smooth since c_y and c_x^{-1} are smooth functions. It follows that the map H_e and its inverse are a smooth. Clearly, $\tilde{x} = H_e^x(x)$ maps the interval $x \in \mathrm{co}\{E_1^x, E_2^x\}$ onto the interval $\tilde{x} \in \mathrm{co}\{\tilde{E}_1^x, \tilde{E}_2^x\}$. Combination of this fact with Eqs. (A.1) and (A.2) yields that $\tilde{q} = H_e(q) \in \tilde{\mathcal{E}}$ if and only if $q \in \mathcal{E}$.

To prove the final statement of the lemma, observe that for each $\delta_1 > 0$, we may choose an $\epsilon > 0$ such that $\|\tilde{E}_i^x - E_i^x\| < \delta_1$, $i = 1, 2$, $\|c(\alpha) - \tilde{c}(\alpha)\| < \delta_1$ and $\|\frac{\partial c}{\partial q} - \frac{\partial \tilde{c}}{\partial q}\| < \delta_1$. The statement $\|\tilde{E}_i^x - E_i^x\| < \delta_1$, $i = 1, 2$, implies that given an arbitrarily small $\delta_2 > 0$ there exists a $\delta_1 > 0$ such that $H_e^x(q) - x < \delta_2$, $H_e^{\dot{x}}(q) - \dot{x} < \delta_2$ for q in a neighbourhood of $\mathcal{E}$. The statement $\|c(\alpha) - \tilde{c}(\alpha)\| < \delta_1$ implies that for each $\delta_3 > 0$ there exists a $\delta_1 > 0$ such that $\|\tilde{\Psi}(x) - \Psi(x)\| < \delta_3$. Hence, for each $\delta > 0$ there exists an $\epsilon > 0$ small enough such that $\|H_e(q) - q\| < \delta$.

The statement $\|\frac{\partial c}{\partial q} - \frac{\partial \tilde{c}}{\partial q}\| < \delta_1$ implies that $|\frac{\partial \Psi}{\partial x} - \frac{\partial \tilde{\Psi}}{\partial \tilde{x}}|$ can be chosen small. Choosing $\epsilon > 0$ small enough one obtains that $|\frac{(\tilde{E}_2^x - \tilde{E}_1^x)}{E_2^x - E_1^x} - 1|$ becomes arbitrarily small. Hence,

$$\frac{\partial H_e}{\partial q} = \begin{pmatrix} \dfrac{(\tilde{E}_2^x - \tilde{E}_1^x)}{E_2^x - E_1^x} & 0 & 0 \\ 0 & \dfrac{(\tilde{E}_2^x - \tilde{E}_1^x)}{E_2^x - E_1^x} & 0 \\ \dfrac{(\tilde{E}_2^x - \tilde{E}_1^x)}{E_2^x - E_1^x}\dfrac{\partial \tilde{\Psi}}{\partial \tilde{x}} - \dfrac{\partial \Psi}{\partial x} & 0 & 1 \end{pmatrix}$$

satisfies the last statement of the lemma. □

In the next result new coordinates are introduced for system A. The equilibrium set $\mathcal{E}$ of A, expressed in these coordinates, will be shown to coincide with the equilibrium set $\tilde{\mathcal{E}}$ of $\tilde{A}$.

Lemma 6. *Consider systems A and $\tilde{A}$ given by (1) with $f, g, \tilde{f}, \tilde{g}$ satisfying (11) for $\epsilon > 0$ sufficiently small. Furthermore, let Assumption 1 be satisfied and let $\frac{\partial g}{\partial y}|_p$ be invertible for all $p \in \mathcal{E}$. The dynamics of system A near the equilibrium set can be described in new coordinates $\bar{q} = \begin{pmatrix} \bar{x} & \dot{\bar{x}} & \bar{y}^T \end{pmatrix}^T$, such that*

$$\begin{aligned} &\ddot{\bar{x}} - \bar{f}(\bar{x}, \dot{\bar{x}}, \bar{y}) \in -F_s \mathrm{Sign}(\dot{\bar{x}}), \\ &\dot{\bar{y}} = \bar{g}(\bar{x}, \dot{\bar{x}}, \bar{y}), \end{aligned} \tag{A.3}$$

where the functions $\bar{f}$ and $\bar{g}$ are smooth, given that ϵ is sufficiently small.

Proof. Let $\bar{q} = H_e(q)$, with $H_e(q)$ given in Lemma 5. Observe that $q \in S_i$ implies $\bar{q} \in S_i$, $i = 1, 2$, when ϵ is small enough, since $\frac{\tilde{E}_2^x - \tilde{E}_1^x}{E_2^x - E_1^x}$ becomes close to one according to the last statement of Lemma 5. Smoothness of $\bar{f}$ and $\bar{g}$ follows from $\dot{\bar{q}} = \bar{F}_i(\bar{q}) = \frac{\partial H_e}{\partial q}\dot{q} = \frac{\partial H_e}{\partial q}F_i(H_e^{-1}(\bar{q}))$ and the fact that H_e and H_e^{-1} are smooth, see Lemma 5, where $\bar{F}_i(\bar{q}) = \begin{pmatrix} \bar{q}_2 & \bar{f}(\bar{q}) + (-1)^{i+1}F_s & \bar{g}^T(\bar{q}) \end{pmatrix}^T$. □

In order to study the structural stability of system (1), the functions f,g and their Jacobian matrices will be important. Hence, the following technical result on $f, g, \bar{f}$ and $\bar{g}$ will be used.

Lemma 7. *Consider systems A and $\tilde{A}$ given by (1) and let the conditions of Lemma 6 be satisfied. Consider a mapping H_e such that $\bar{q} = H_e(q)$ yields the coordinates given in Lemma 6. For each $\delta > 0$ there exists an $\epsilon > 0$ such that (11) implies that, firstly, $\|\frac{\partial g}{\partial y} - \frac{\partial \bar{g}}{\partial \bar{y}}\| < \delta$ and secondly, $\|\frac{\partial F_i}{\partial q}|_{E_i} - \frac{\partial \bar{F}_i}{\partial \bar{q}}|_{\bar{E}_i}\| < \delta$, $i = 1, 2$, where $\bar{F}_i(\bar{q}) = \begin{pmatrix} \bar{q}_2 & \bar{f}(\bar{q}) + (-1)^{i+1}F_s & \bar{g}^T(\bar{q}) \end{pmatrix}^T$.*

Proof. Given $\delta_1 > 0$, Lemma 5 states that there exists an $\epsilon > 0$ such that (11) implies $\|\frac{\partial H_e}{\partial q} - I\| < \delta_1$. It follows that $\frac{\partial H_e^{-1}}{\partial q}$ becomes close to identity as well, such that for each $\delta_2 > 0$, there exists an $\epsilon > 0$ such that (11) implies, firstly, that $\|\frac{\partial H_e}{\partial q} - I\| < \delta_2$ and secondly, $\|\frac{\partial H_e^{-1}}{\partial q} - I\| < \delta_2$. Hence, the properties of $\bar{g}$ and $\bar{F}$ follow from $\frac{\partial \bar{F}_i}{\partial \bar{q}}|_{\bar{E}_i} = \frac{\partial^2 H_e}{\partial q \partial \bar{q}} F_i(H_e^{-1}(\bar{E}_i)) + \frac{\partial H_e}{\partial q}\frac{\partial F_i}{\partial q}\frac{\partial H_e^{-1}}{\partial \bar{q}}(\bar{E}_1)$, where $F_i(H_e^{-1}(\bar{E}_i)) = F_i(E_i) = 0$. □

We now obtained a coordinate transformation H_e, which can be applied to guarantee that the equilibrium sets of A and $\tilde{A}$ coincide. Hence, assuming that the equilibrium sets of A and $\tilde{A}$ coincide will not introduce a loss of generality. The properties of the dynamical equation (1) that will be used in the proof of Theorem 1 are not changed by this coordinate transformation. Note, that a similar argument is used to study perturbations of a hyperbolic equilibrium point for a smooth dynamical system; see e.g. [39].

Appendix B. Proof of Theorem 1

To prove Theorem 1, first the structural stability of the sliding dynamics given by (8) is investigated. Subsequently, it is shown that a topological map, i.e. a homeomorphism satisfying the conditions given in Definition 1, can be extended orthogonal to this boundary if it exists for the sliding trajectories on Σ. An eigenvalue λ will be considered critical if $\text{real}(\lambda) = 0$.

Lemma 8. *Let system (1) satisfy Assumption 1 and let $F_s > 0$. If $\frac{\partial g}{\partial y}|_p$ does not have critical eigenvalues for all $p \in \mathcal{E}$, then for any closed set $J \subset I$ the sliding trajectories of system (1) in a neighbourhood $N(J) \subset \Sigma$ of J are structurally stable for perturbations of f and g.*

Proof. We consider a system A given by (1) and a perturbed system $\tilde{A}$ described by (1) with $\tilde{f}, \tilde{g}$ satisfying (11) for sufficiently small $\epsilon > 0$. By Lemma 7, we may assume without loss of generality that the equilibrium set of A and $\tilde{A}$ coincide.

The sliding solutions of A are described by (8). Restricting this dynamics to the boundary Σ, one obtains:

$$\begin{aligned} \dot{x} &= 0, \\ \dot{y} &= g(x, 0, y), \end{aligned} \tag{B.1}$$

for system A and

$$\begin{aligned} \dot{x} &= 0, \\ \dot{y} &= \tilde{g}(x, 0, y), \end{aligned} \tag{B.2}$$

for system $\tilde{A}$. Note that $\epsilon > 0$ can be chosen such that $\frac{\partial \tilde{g}}{\partial y}$ is arbitrarily close to $\frac{\partial g}{\partial y}$. Hence, the Jacobian matrix $\frac{\partial g}{\partial y}|_p$ for $p \in \mathcal{E}$ has the same number of positive and negative eigenvalues as $\frac{\partial \tilde{g}}{\partial y}|_p$. This implies that trajectories of $\dot{y} = \tilde{g}(x, 0, y)$ are locally topologically equivalent near p to the trajectories dynamics of $\dot{y} = g(x, 0, y)$; cf. Theorem 5.1 of [39, page 68]. From the reduction theorem, cf. [40, page 15], we conclude that (B.1) and (B.2) are topologically equivalent. □

The importance of the topological nature of sliding trajectories will be shown using the following lemma. To prove this lemma, the following notation is used. Let $q(t) = \varphi(t, q_0)$ denote a solution of system A given by (1) in the sense of Filippov with initial condition $q(0) = \varphi(0, q_0) = q_0$. Furthermore, let $q(t) = \varphi^i(t, q_0)$, $i = 1, 2$, denote a trajectory of $\dot{q} = F_i(q)$ satisfying $q(0) = \varphi^i(0, q_0) = q_0$. Analogously, the functions $\tilde{\varphi}(\cdot,\cdot)$, $\tilde{\varphi}^i(\cdot,\cdot)$, $i = 1, 2$, are defined for the perturbed system $\tilde{A}$. We note that this notation can be used for trajectories in reverse time when $t < 0$.

Lemma 9. *Let system (1) satisfy Assumption 1 and let $F_s > 0$. For every interior point $p \in I$ there exists a neighbourhood $N(p)$ and a finite $\tau > 0$ such that any trajectory with an initial condition in $N(p)$ arrives in Σ^s at time $t \in [0, \tau]$.*

Proof. This proof follows the lines as described by Filippov; see [19, page 262]. Consider an interior point $p \in I$, and observe that $I \subset \Sigma^s$. Trajectories near p on Σ^s trivially satisfy the lemma for $t = 0$. Hence, we restrict our attention to trajectories in S_1 near p. The trajectories in S_2 can be handled analogously. The interior point $p \in I \subset \Sigma^s$, such that (7) implies $L_{F_1}h(p) > 0$. Since F_1 is smooth, there exist an $\delta > 0$ and neighbourhood $N_a(p)$ of p such that $L_{F_1}h(q) > \delta, \forall q \in N_a$. Since F_1 is smooth, for all $q \in N_a(p) \cap \Sigma$ there exists a unique trajectory $\varphi^1(t, q)$ of $\dot{q} = F_1(q)$ that satisfies $\varphi^1(0, q) = q$. By $L_{F_1}h(q) > \delta$ there exists a $\tau > 0$ such that $\varphi^1(t, q) \in S_1$, $\forall t \in [-\tau, 0)$. Hence, the trajectory $\varphi^1(t, q)$ coincides with a trajectory of (1) on this time interval. From Theorem 3, [19, page 128], we conclude that these trajectories form a one-sided neighbourhood $N_1(p)$ of p. By studying the trajectories in S_2, analogously we find a one-sided neighbourhood $N_2(p)$. Since there exists a neighbourhood $N(p) \subset \Sigma^s \cup N_1(p) \cup N_2(p)$, the lemma is proven. □

From this lemma it follows that the qualitative nature of trajectories near interior points $p \in I$ can be described as follows. According to Lemma 9, trajectories arrive at the discontinuity surface in finite time. Subsequently, these trajectories are described by the sliding vector field. Using this property, a topological map defined for sliding trajectories on Σ can be extended towards a neighbourhood of this boundary, which is formalised in the next lemma.

Lemma 10. *Consider two systems A and $\tilde{A}$ and let there exist a topological map $H_\Sigma : U \mapsto \tilde{U}$, where $U \subset \Sigma$ and $\tilde{U} \subset \tilde{\Sigma}$. If $(-1)^i L_{F_i}h(q) < 0$, $\forall q \in U$, and $(-1)^i L_{\tilde{F}_i}h(\tilde{q}) < 0$, $\forall \tilde{q} \in \tilde{U}$, for $i = 1$ or $i = 2$, then one can extend the topological map H_Σ towards S_i such that H_Σ is defined in a closed n-dimensional subset of $U \cup S_i$ that contains U.*

Proof. To prove the lemma, we exploit the assumption that $L_{F_1}h(q) > 0$, $\forall q \in U$ and $L_{\tilde{F}_1}h(\tilde{q}) > 0$, $\forall \tilde{q} \in \tilde{U}$. The case $L_{F_2}h(q) < 0$, $\forall q \in U$ and $L_{\tilde{F}_2}h(\tilde{q}) < 0$, $\forall \tilde{q} \in \tilde{U}$ can be handled analogously. We observe that $L_{F_1}h(q) > 0$, $\forall q \in U$ implies that for any $q \in U$, there exists a unique trajectory of system A that satisfies $\varphi(0, q) = q$ and $\varphi(t, q) \in S_1$ for $t \in [T_1, 0)$, with $T_1 < 0$. Choosing $\tilde{q} = H_\Sigma(q)$, $L_{\tilde{F}_1}h(H_\Sigma(q)) > 0$ analogously implies that there exists a unique trajectory of $\tilde{A}$ such that $\tilde{\varphi}(0, H_\Sigma(q)) = H_\Sigma(q) \in \tilde{U}$ and $\tilde{\varphi}(t, H_\Sigma(q)) \in \tilde{S}_1$ for $t \in [T_2, 0)$, where $T_2 < 0$. Introducing $T = \max(T_1, T_2)$ yields $\varphi(t, q) \in S_1$ and $\tilde{\varphi}(t, H_\Sigma(q)) \in \tilde{S}_1$, $\forall t \in [T, 0)$, $\forall q \in U$.

From Theorem 3, [19, page 128], we conclude that the union of these trajectories form compact, connected n-dimensional sets V and $\tilde{V}$ containing U and $\tilde{U}$. Hence, there exist a unique map $\psi : V \mapsto [T, 0] \times U$ such that $(\tau, \rho) = \psi(q)$ with inverse $q = \varphi(\tau, \rho)$. Here, ρ denotes the first point where the trajectory with initial condition q crosses Σ, the time lapse is denoted $-\tau$. Since $(-1)^i L_{F_i}h(q) < 0$, $\forall q \in U$, there exists a unique trajectory $\varphi^i(t, \rho)$ of $\dot{q} = F_i(q)$ with initial condition $\varphi(0, \rho) = \rho \in U$ at time $t = 0$ that crosses Σ non-tangential (i.e. transversal) at time $t = 0$ and satisfies $\varphi(t, q) \in S_i$, $\forall t \in (T, 0)$. For this time interval, the trajectory φ^i of $\dot{q} = F_i(q)$ coincides with the trajectory φ of A. Hence, we observe that ψ is continuous and unique; cf. [26, page 242].

Now, we define $H_\Sigma(q)$ for $q \notin \Sigma$ as $H_\Sigma(q) = \tilde{\varphi}(\tau, H_\Sigma(\rho))$, where $(\tau, \rho) = \psi(q)$. We observe that $H_\Sigma : V \mapsto \tilde{V}$ satisfies the conditions of the lemma, and maps trajectories of A onto trajectories of $\tilde{A}$. □

The following lemma gives sufficient conditions for the structural stability of trajectories near interior points of the equilibrium set. Since trajectories near the endpoints E_1 and E_2 of the equilibrium set are not considered, the following lemma is restricted to arbitrarily closed subsets of I.

Lemma 11. *Let system (1) satisfy Assumption 1 and let $F_s > 0$. If $\frac{\partial g}{\partial y}|_p$ does not have critical eigenvalues for all $p \in \mathcal{E}$, then for any closed set $J \subset I$ the trajectories of system (1) in a neighbourhood $N(J)$ of J are structurally stable for perturbations of f and g.*

Proof. Consider system A described by (1) and let the perturbed version be denoted by $\tilde{A}$. Under the conditions given in the lemma, the result of Lemma 8 implies that there exists a topological map H_Σ in a set $\bar{N}(J) \subset \Sigma$ containing J, that maps trajectories of A onto trajectories of $\tilde{A}$. Observing that $\bar{N}(J) \subset \Sigma^s$, one can apply Lemma 10, which proves that the map H_Σ can be extended to subsets of S_1 and S_2. In this manner, a topological map is obtained from a neighbourhood of J to a neighbourhood of $H_\Sigma(J)$, which proves the lemma, see Definition 2.

Under the conditions given in the foregoing lemma, we conclude that no changes can occur in the topological nature of trajectories around interior points of the equilibrium set. Hence, we are able to prove Theorem 1.

Proof of Theorem 1. Theorem 1 is proven by contradiction. Suppose there exists a system $A(\mu)$ smoothly depending on parameter μ, that undergoes a local bifurcation at parameter $\mu = 0$ which does not occur at the endpoints. Furthermore, we use the assumption in the theorem that $\frac{\partial g}{\partial y}|_p$ does not have critical eigenvalues for system $A(0)$. A sufficiently small change of μ near 0 can be chosen such that systems $A(0)$ and $A(\mu)$ satisfy (11) for arbitrarily small ϵ. Reversing the direction of time if necessary, we may assume $F_s > 0$ such that Lemma 11 can be applied. This lemma contradicts the occurrence of a local bifurcation of the equilibrium set at an interior point. □

Appendix C. Proof of Theorem 2

In this section, Theorem 2 is proven. First, we study the trajectories in the neighbourhood of an individual endpoint E_k with $k = 1$ or $k = 2$, where the eigenvalues of A_k are complex. Subsequently, endpoints are studied where A_k has real eigenvalues. Recall that a topological map is a homeomorphism satisfying the conditions given in Definition 1.

Lemma 12. *Consider a planar system A given by (12) satisfying Assumptions 1 and 2, where A_k, with $k = 1$ or $k = 2$, has complex eigenvalues $\lambda = \alpha \pm \imath\omega$, $\omega \neq 0$. Furthermore, let $\tilde{A}$ be a perturbed system satisfying (11) with sufficiently small $\epsilon > 0$. There exist a topological map H_c and neighbourhoods $N(E_k)$ of E_k and $\tilde{N}(\tilde{E}_k)$ of $\tilde{E}_k$ such that $H_c : N(E_k) \mapsto \tilde{N}(\tilde{E}_k)$.*

Proof. By Lemma 7, we may assume that the equilibrium sets of A and $\tilde{A}$ coincide. From (12) it follows, together with $F_s > 0$ of Assumption 2, that the equilibrium sets $\mathcal{E}$ and $\tilde{\mathcal{E}}$ coincide with stable sliding motion, i.e. $\mathcal{E} = \Sigma^s$ and $\tilde{\mathcal{E}} = \tilde{\Sigma}^s$.

In this proof we consider the case $k = 1$, the proof for $k = 2$ can be derived analogously. Let the real matrix P be given by the real Jordan decomposition of A_1 given by $A_1 = PJP^{-1}$, where $J = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}$. Hence, the linear system $\dot{w} = A_1 w$ can be represented by new coordinates r, θ such that $\bar{w}_1 = r\cos(\theta)$ and $\bar{w}_2 = r\sin(\theta)$, with $\begin{pmatrix} \bar{w}_1 & \bar{w}_2 \end{pmatrix}^T = P^{-1}w$. In these coordinates, the dynamics $\dot{w} = A_1 w$ yields $\dot{r} = \alpha r$ and $\dot{\theta} = \omega$. Hence, all trajectories of this system encircle the origin, and cross every line through the origin every $T = \frac{\pi}{\omega}$ time units. If we let $w = q - E_1$, then $\dot{w} = A_1 w$ serves as a linear approximation of $\dot{q} = F_1(q)$ near E_1. Using the same coordinates r and θ, we obtain $\dot{\theta} = \omega + \bar{\omega}(r, \theta)$,

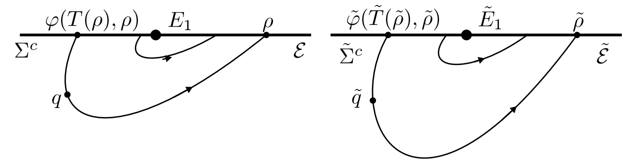

Fig. C.8. Sketch of trajectories near E_1 for a 2-dimensional system when A_1 has complex eigenvalues. The left panel shows trajectories of A, the right panel shows trajectories of a perturbed system $\tilde{A}$.

where $\bar{\omega}$ is determined by the error introduced by the linearisation. Hence, the function $\bar{\omega}$ is smooth and satisfies $\bar{\omega}(r, \theta) = \mathcal{O}(r^2)$. For each $\delta > 0$, there exists a small enough neighbourhood $N_1(E_1)$ of E_1 such that $\dot{\theta} \in [\omega - \delta, \omega + \delta]$ and $\dot{r} \in [\alpha r - \delta, \alpha r + \delta]$.

This implies that there exists a neighbourhood $N_2(E_1) \subset N_1(E_1)$ of E_1, such that trajectories of $\dot{q} = F_1(q)$ in $N_2(E_1)$ cross every line through E_1 in a transversal manner; see Fig. C.8. Hence, for each $q \in \Sigma^s \cap N_2(E_1)$ there exist a $T(q)$ near $-\frac{\pi}{\omega}$ such that $\varphi^1(T(q), q) \in \Sigma^c$, where $\varphi^1(t, q)$ denotes the trajectory of $\dot{q} = F_1(q)$ with initial condition $\varphi^1(0, q) = q$ at time $t = 0$. The function $T(q)$ is continuous; see [26, page 242]. Since $L_{F_1}h(q) > 0$ for $q \in \Sigma^s$, the trajectory $\varphi^1(t, q)$ crosses Σ^s from S_1, such that $\varphi^1(t, q) \in S_1$, $\forall t \in (T(q), 0)$. By Filippov's solution convention, the trajectory $\varphi^1(t, q)$ coincides with a trajectory of (12) for $t \in [T(q), 0]$. Furthermore, any point $q \in S_1 \cap N_2(E_1)$ can be described with the coordinates (τ, ρ), $\rho \in \Sigma^s$, $\tau \in (T(\rho), 0)$, such that $q = \varphi^1(\tau, \rho)$.

Now, consider a perturbed system $\tilde{A}$ satisfying (11) with ϵ sufficiently small. This implies the eigenvalues of $\tilde{A}_1$ are arbitrarily close to the eigenvalues of A, and hence are complex. Analogous to the reasoning given in the foregoing paragraph for system A, one can show that there exists a continuous function $\tilde{T}(\tilde{\rho})$ for $\tilde{\rho} \in \tilde{\Sigma}^s$, such that $\tilde{\varphi}^1(\tilde{T}(\tilde{\rho}), \tilde{\rho}) \in \tilde{\Sigma}^c$ and any point $\tilde{q} \in \tilde{S}_1 \cap \tilde{N}_2(\tilde{E}_1)$ can be described with the coordinates $(\tilde{\tau}, \tilde{\rho})$, $\tilde{\tau} \in (\tilde{T}(\tilde{\rho}), 0)$, such that $\tilde{q} = \tilde{\varphi}^1(\tilde{\tau}, \tilde{\rho})$.

Let H_{c1} map points $(\tau, \rho) \to (\tilde{\tau}, \tilde{\rho}) = (\frac{\tilde{T}(\rho)}{T(\rho)}\tau, \rho)$. This map and its inverse are locally continuous since $T(\rho)$ and $\tilde{T}(\rho)$ are nonzero, and the functions $T(\cdot)$, $\tilde{T}(\cdot)$ are continuous.

Now, let H_c map $\varphi^1(\tau, \rho) \to \tilde{\varphi}^1(\tilde{\tau}, \tilde{\rho})$, where $(\tilde{\tau}, \tilde{\rho}) = H_{c1}(\tau, \rho)$. This map is continuous away from the point E_1 since H_{c1} is continuous and both φ^1 and $\tilde{\varphi}^1$ are trajectories of a smooth system, hence φ^1 and $\tilde{\varphi}^1$ are continuous.

The domain of definition of H_c is extended from S_1 towards Σ as follows. For points $q \in \Sigma$ we choose a sequence $\{q_i\}$ with $\lim_{i\to\infty} q_i = q \in \Sigma$ and $q_i \in S_1$ and define $H_c(q) = \lim_{i\to\infty} H_c(q_i)$. In this manner, the domain of H_c becomes $(S_1 \cup \Sigma) \cap N_2(E_1)$.

Continuity of H_c is trivial away from the point E_1. Now, continuity of H_c at E_1 is proven. Consider two arbitrary sequences $\{q_i\} \in S_1$ and $\{\bar{q}_j\} \in S_1$ with $\lim_{i\to\infty} q_i = \lim_{j\to\infty} \bar{q}_j = E_1$. The two sequences correspond to different coordinates $\{\tau_i, \rho_i\}$ and $\{\bar{\tau}_j, \bar{\rho}_j\}$. We observe that both $\lim_{i\to\infty} \rho_i = E_1$ and $\lim_{j\to\infty} \bar{\rho}_i = E_1$, whereas the limits of the τ and $\bar{\tau}$-sequences may differ. However, $\tilde{E}_1 = E_1$ is an equilibrium of $\dot{\tilde{q}} = \tilde{F}_1(\tilde{q})$, such that $\tilde{\varphi}^1(t, \tilde{E}_1) = \tilde{E}_1$, is independent on t. Hence, we obtain $\tilde{E}_1 = H_c(E_1) = \lim_{i\to\infty} H_c(q_i) = \lim_{j\to\infty} H_c(\bar{q}_j)$. Continuity of H_c at E_1 is proven.

At E_1 one finds $F_1(E_1) = 0$, such that $L_{F_2}h(E_1) = -2F_s$; see (4) and (5). Hence, according to Lemma 10 the domain of definition of H_c can be extended towards a neighbourhood of E_1 in S_2, such that $H_c : N(E_1) \mapsto \tilde{N}(\tilde{E}_1)$, where $N(p)$ denotes a neighbourhood of p and $\tilde{N}(\tilde{p})$ denotes a neighbourhood of $\tilde{p}$. The map H_c is a topological map. □

Lemma 12 proves that the trajectories near an endpoint E_k, $k = 1$ or $k = 2$ are structurally stable when the matrix A_k has complex eigenvalues, such that all trajectories in S_k near E_k will encircle this point and either enter Σ^s, or cross the boundary Σ^c in finite time.

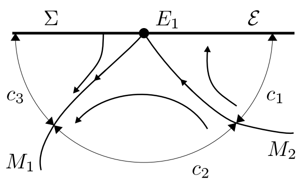

Fig. C.9. Sketch of trajectories near E_1 for a 2-dimensional system when A_1 has two real eigenvalues. The left panel shows trajectories of A, the right panel shows trajectories of a perturbed system $\tilde{A}$.

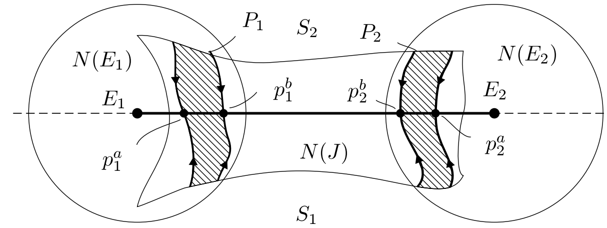

Fig. C.10. Schematic construction of domains P_1 and P_2, which are shown hatched.

In the following lemma, structural stability is studied of trajectories in the neighbourhood of an endpoint E_k when the matrix A_k has real eigenvalues. Hereto, separatrices of this system, consisting of trajectories that converge asymptotically to the endpoint E_k, are introduced as follows.

In case $A_k,\ k = 1$ or $k = 2$, has real nonzero eigenvalues, Assumption 2 implies that the eigenvectors v_j of A_k satisfy $\nabla h v_j \neq 0$, since $A_k = \begin{pmatrix} 0 & 1 \\ \frac{\partial f}{\partial x}|_{E_k} & \frac{\partial f}{\partial \dot{x}}|_{E_k} \end{pmatrix}$. To study the case where A_k has real eigenvalues, separatrices of the system are studied. Assume A_k has nonzero, real eigenvalues and let H_{HG} denote the topological map that maps trajectories of (12) in S_k to trajectories of $\dot{q} = A_k q$, which exist according to the Hartman–Grobman theorem. Given an eigenvalue λ_j, let v_j denote the unique corresponding unit eigenvector of A_k that points towards S_i. The set $\bar{M}_j := \{q \in \mathbb{R}^2 : q = c v_j,\ c \in (0, \infty)\},\ j = 1, 2$, is invariant for $\dot{q} = A_k q$, such that the set $M_j := \{E_k\} + H_{HG}^{-1}(\bar{M}_j)$ is invariant for $\dot{q} = F_k(q - E_k)$. Since the set M_j consist of a single trajectory of the smooth differential equation $\dot{q} = F_k(q)$, M_j is a smooth curve. The invariant manifold M_j is a separatrix for system (12) that is tangent to $\bar{M}_j$ at E_k. If A_k has two distinct real eigenvalues, the separatrices M_1 and M_2 correspond to the eigenvalues λ_1 and λ_2 of A_k. Note that M_1 and M_2 coincide with the stable and unstable manifold of E_k positioned in S_k if the eigenvalues of A_k satisfy $\lambda_1 < 0 < \lambda_2$. For perturbed systems $\tilde{A}$, the separatrices $\tilde{M}_1$ and $\tilde{M}_2$ are defined analogously.

Lemma 13. *Consider a planar system A given by (12) satisfying Assumptions 1 and 2, where A_k, with $k = 1$ or $k = 2$, has real, nonzero eigenvalues. Furthermore, let $\tilde{A}$ be a perturbed system satisfying (11) with sufficiently small $\epsilon > 0$. There exist a topological map H_r and neighbourhoods $N(E_k)$ of E_k and $\tilde{N}(\tilde{E}_k)$ of $\tilde{E}_k$ such that $H_r : N(E_k) \mapsto \tilde{N}(\tilde{E}_k)$.*

Proof. In this proof we consider the case $k = 1$; the proof for $k = 2$ can be derived analogously. Since the eigenvalues and eigenvectors of a real, nonsingular matrix are continuous functions of parameters, the eigenvalues and eigenvectors of $\tilde{A}_1$ are close to those of A_1. Hence, $\tilde{A}_1$ has real, nonzero eigenvalues, and separatrices $\tilde{M}_1$ and $\tilde{M}_2$ of $\tilde{A}$ are locally close to M_1 and M_2. From (12) it follows, together with $F_s > 0$ of Assumption 2, that the equilibrium sets $\mathcal{E}$ and $\tilde{\mathcal{E}}$ near the points E_1 or $\tilde{E}_1$ coincide with stable sliding motion, hence $\mathcal{E}$ and $\tilde{\mathcal{E}}$ coincide with Σ^s and $\tilde{\Sigma}^s$, respectively.

For the system A the separatrices M_1 and M_2 partition the domain $N(E_1) \cap S_1$ into three sectors c_1, c_2, c_3; see Fig. C.9. The index of c_1 is chosen such that $\mathcal{E}$ is a subset of the boundary of c_1 and the boundary of c_3 contains Σ^c, as shown in Fig. C.9. Similar, we partition $\tilde{N}(\tilde{E}_1)$ into three domains $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$, bounded by the manifolds $\tilde{M}_1, \tilde{M}_2, \tilde{\Sigma}$ and $\tilde{E}_1$.

Trajectories in $c_2 \cup M_1 \cup M_2$ of A, are described by $\dot{q} = F_1(q)$ and trajectories of $\tilde{A}$ in $c_2 \cup M_1 \cup M_2$ are described by $\dot{\tilde{q}} = \tilde{F}_1(\tilde{q})$. Near E_1, resp E_2, these trajectories are locally equivalent according to Lemma 9 and 10 of [41, page 306–307]. Hence, there exist a topological map H_r^2 mapping $c_2 \cup M_1 \cup M_2$ onto $\tilde{c}_2 \cup \tilde{M}_1 \cup \tilde{M}_2$.

For a sufficiently small neighbourhood $N(E_1)$, by Lemma 3 of [19, page 194], there exist topological maps H_r^1 and H_r^3 from trajectories of A in $c_1 \cap N(E_1)$ and $c_3 \cap N(E_1)$, respectively, onto trajectories of $\tilde{A}$ in sectors $\tilde{c}_1 \cap \tilde{N}(\tilde{E}_1)$ and $\tilde{c}_3 \cap \tilde{N}(\tilde{E}_1)$, respectively. According to [19, page 196], the topological maps H_r^1 and H_r^3 can be chosen to coincide at separatrices M_1, M_2 with H_r^2. In this manner, we obtain a topological map $H_r : N(E_1) \cap (\Sigma \cup S_1) \to \tilde{N}(\tilde{E}_1) \cap (\tilde{\Sigma} \cup \tilde{S}_1)$.

At E_1 one finds $F_1(E_1) = 0$, such that $L_{F_2} h(E_1) = -2F_s$; see (4) and (5). Hence, Lemma 10 is applied. The domain of definition of H_r is extended towards a subset of S_2 such that H_r maps $N(E_1)$ onto $\tilde{N}(\tilde{E}_1)$. □

Lemmas 12 and 13 guarantee structural stable properties for system (12) in neighbourhoods of the endpoints of an equilibrium set under certain conditions on the linearised dynamics around these endpoints. These results are combined with Theorem 1, to prove Theorem 2.

Proof of Theorem 2. Let $\tilde{A}$ be an arbitrary system satisfying (11) for ϵ sufficiently small. By Lemma 7, we may assume that the equilibrium sets of A and $\tilde{A}$ coincide. According to Assumption 2, the eigenvalues of A_1 and A_2 are either complex or real, nonzero and distinct. In the first case, Lemma 12 guarantees that there exists a topological map $H_k,\ k = 1, 2$, from a neighbourhood $N(E_k)$ of E_k to a neighbourhood of $\tilde{E}_k$; in the case of real eigenvalues this is guaranteed by Lemma 13. Now, in the neighbourhood $N(E_1)$ we select two arbitrary points p_1^a, p_1^b such that $p_1^a, p_1^b \in I \cap N(E_1)$ and $0 < |p_1^a - E_1| < |p_1^b - E_1|$; see Fig. C.10. Analogously, we select two points p_2^a, p_2^b. Choosing $J = \text{co}\{p_1^a, p_2^a\}$ yields $J \subset I$, such that Lemma 11 implies there exists a topological map H_J defined in a neighbourhood $N(J)$ of J. According to [41, page 196], the topological maps $H_k,\ k = 1, 2$, can be chosen to coincide with H_J at the equilibrium points, such that $H_k(p) = H_J(p),\ \forall p \in \text{co}\{p_k^a, p_k^b\}$ for $k = 1, 2$.

To obtain a topological map that is continuous and coincides with H_k near $E_k,\ k = 1, 2$, and coincides with H_J for points further away from these endpoints, transition regions P_1 and P_2 are introduced, given by $P_k(J) := \{q \in N(J) \cap N(E_k) : \lim_{t\to\infty} \varphi(q, t) \in \text{co}\{p_k^a, p_k^b\}\},\ k = 1, 2$; see Fig. C.10. In these regions, new topological maps are introduced that connect H_J and $H_k,\ k = 1, 2$, in a continuous fashion.

In this manner, a topological map is constructed in a neighbourhood of $\mathcal{E}$, which is constructed as follows. One can select a subset $N'(J) \subset N(J)$ containing J, and $N'(E_k) \subset N(E_k),\ k = 1, 2$, such that $N'(E_1) \cup N'(E_2) \cup P_1 \cup P_2 \cup N'(J)$ contains a neighbourhood of $\mathcal{E}$, the interiors of the sets $N'(E_k),\ k = 1, 2$, and $N'(J)$ have an empty intersection and the domains $N'(E_k),\ k = 1, 2$, and $N'(J)$ intersect with P_k only at a one-dimensional set; cf. Fig. C.11. For $q \in P_k,\ k = 1, 2$, we will construct a topological map H_{P_k} that coincides with H_k for $q \in N'(E_k)$ and coincides with H_J for $q \in N'(J)$.

We will now proceed to construct the map H_{P_1} that connects H_1 and H_J in a continuous fashion. Analogously, a map H_{P_2} can be constructed.

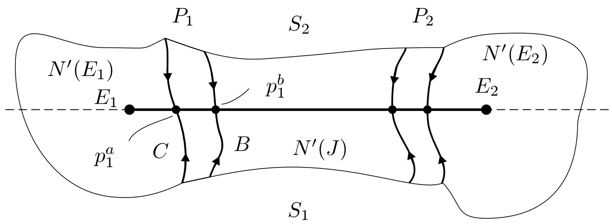

Fig. C.11. Schematic construction of a neighbourhood of $\mathcal{E}$.

Since H_J and H_1 coincide for $q \in P_1 \cap J = \text{co}\{p_1^a, p_1^b\}$, we need to construct a map H_{P_1} that, firstly, coincides with H_J at the intersection of the closures of $N'(J)$ and P_1 and secondly, coincides with H_1 at the boundary between $N'(E_1)$ and P_1. Let the function H_{P_1} coincide with H_J for $q \in P_1 \cap J$. Since $P_1 = (P_1 \cap S_1) \cup (P_1 \cap J) \cup (P_1 \cap S_2)$, firstly we will construct the map H_{P_1} for points in $P_1 \cap S_1$ such that it coincides with H_J at the boundary between P_1 and $N'(J)$, which is denoted with B, and coincides with H_1 at the boundary between P_1 and $N'(E_1)$, which is denoted with C; cf. Fig. C.11.

Both maps H_1 and H_J satisfy $H_1(q) \in \tilde{S}_1$ and $H_J(q) \in \tilde{S}_1$ when $q \in S_1$. Hence, we will introduce new coordinates in $\tilde{S}_1$. Similar to the proof of Lemma 10, for points $\tilde{q} \in \tilde{S}_1$ we introduce the new coordinates (τ, ρ) given by $\psi : P_1 \to [T, 0] \times [p_1^a, p_1^b]$ with $T < 0$, such that $(\tau, \rho) = \psi(\tilde{q})$, $\tilde{\varphi}^1(\tau, \rho) = \tilde{q}$ and $\rho \in [p_1^a, p_1^b] \subset J$, where $\tilde{q}(t) = \tilde{\varphi}^1(t, \tilde{q}_0)$ is a trajectory of $\dot{\tilde{q}} = \tilde{F}_1(\tilde{q})$ with initial condition $\tilde{q}(0) = \tilde{q}_0$. For $q \in P_1 \cap S_1$, both the maps $H_1(q)$ and $H_J(q)$ can be expressed in the coordinates (τ, ρ), such that $(H_1^\tau, H_1^\rho) = \psi(H_1(q))$ and $(H_J^\tau, H_J^\rho) = \psi(H_J(q))$.

The functions $H_1^\rho(q)$ and $H_J^\rho(q)$ denote the position of the first crossing with the boundary Σ of trajectories of system A with initial conditions q. At $\Sigma \cap P_1$, the maps H_1 and H_J coincide since $\Sigma \cap P_1$ is part of the equilibrium set. Since both H_1 and H_J map trajectories onto trajectories and the maps coincide for $q \in \Sigma \cap P_1 = \text{co}\{p_1^a, p_1^b\}$, we find that $H_1^\rho(q) = H_J^\rho(q),\ \forall q \in P_1$.

Recall that the map H_{P_1} should coincide with H_J at curve B and with H_1 at curve C. For the variable τ, we introduce a linear interpolation $H_{P_1}^\tau$ given by $H_{P_1}^\tau(q) = H_1^\tau(q) + \frac{H_1^\rho(q) - p_1^a}{p_1^b - p_1^a}(H_J^\tau(q) - H_1^\tau(q))$, which clearly is a function that blends H_1^τ and H_J^τ in set $P_1 \cap S_1$. Transforming $(H_1^\rho(q), H_{P_1}^\tau(q))$ to the coordinates $\tilde{q}$, a topological map H_{P_1} is defined in $P_1 \cap S_1$. Expressed in (ρ, τ) coordinates, it immediately follows that firstly, H_{P_1} coincides with H_J at curve B, secondly, it coincides with H_1 at curve C and finally, $H_{P_1}(p) = H_1(p) = H_J(p),\ \forall p \in P_1 \cap J$. Analogously, H_{P_1} is defined in $P_1 \cap S_2$, such that $H_{P_1}(q)$ is defined for all $q \in P_1$. The map H_{P_2} can be constructed with the same procedure.

Now, we define

$$H(q) = \begin{cases} H_1(q), & q \in N'(E_1), \\ H_J(q), & q \in N'(J), \\ H_2(q), & q \in N'(E_2), \\ H_{P_1}(q), & q \in P_1, \\ H_{P_1}(q), & q \in P_2, \end{cases} \tag{C.1}$$

which is a topological map, proving the theorem. □

Appendix D. Proof of Lemma 3

Proof of Lemma 3. The existence of a minimum distance between interior points and a limit cycle is trivial, since trajectories close to the interior points are attracted to the equilibrium set; cf. Lemma 9. Hence, we need to prove that any limit cycle remains away from the endpoints E_1 and E_2. In this proof we will show that the limit cycle does not come close to the equilibrium point E_1, the second case can be excluded analogously. Without loss of generality, we assume $F_s > 0$. For the sake of contradiction, we assume that a limit cycle γ comes close to E_1 and has finite period T_γ. Since the vector field F_2 is pointing towards S_1 in a neighbourhood of E_1, the limit cycle should leave the neighbourhood of E_1 in direction of S_1. Since the dynamics in S_1 is described by the smooth differential equation $\dot{q} = F_1(q)$, we observe that $\|\dot{q}\|$ becomes arbitrarily small near E_1. Hence, a trajectory on γ will spend an arbitrarily long period of time in the neighbourhood of E_1. This implies that the period time of the closed orbit γ becomes arbitrarily long, yielding a contradiction. The lemma is proven. □

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