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Estimation of basins of attraction for controlled systems with input saturation and time-delays

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HIGHLIGHTS

- The basin of attraction of saturated control systems with time-delay is described.
- Piecewise quadratic Lyapunov–Krasovskii functionals are proposed.
- Sublevel sets of these functionals provide basin of attraction estimates.
- The functional is not necessarily positive definite.
- A numerical procedure is presented that attains effective estimates in examples.

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ABSTRACT

Basins of attraction are instrumental to study the effect of input saturation in control systems, as these sets characterise the initial conditions for which the control strategy induces attraction to the desired state. In this paper, we describe these sets when the open-loop system is exponentially unstable and the system is controlled by actuators with both constant time-delays and saturation. Estimates of the basin of attraction are provided and the allowable time-delay in the control loop is determined with a novel piecewise quadratic Lyapunov–Krasovskii functional that exploits the piecewise affine nature of the system. As this approach leads to sufficient, but not to necessary conditions for attractivity, we present simulations for two examples to show the applicability of the results.

1. Introduction

Input saturations and delays occur in virtually all control systems in mechanical, chemical and electric engineering. However, in the control design process, the non-linear effect of saturations is often ignored, and most studies including time-delays in their analysis consider linear systems. We study the effects of both input saturation and constant time-delays on the closed-loop dynamics.

We focus on linear systems controlled by actuators with saturation and a common delay. Restricting our attention to static controllers, "windup"-type problems, as addressed in [1] for the delay-free case, are excluded. We present a method to estimate the basin of attraction for closed-loop systems with input saturation and delays. This is the set of initial conditions for which the controller achieves convergence to the origin. Consequently, the basin of attraction is instrumental in accessing the effect of the saturation and delays.

In the literature, basins of attraction for smooth (closed-loop) systems without time-delays are well-understood, and, under some technical conditions, the geometry of these basins of attraction can be approximated arbitrarily closely with the sublevel sets of polynomial Lyapunov functions, cf. [2]. For control systems with saturation and without delays, in [3], both performance of the controlled system and its basins of attraction are described. In [4], piecewise quadratic Lyapunov functions are presented and in [5,6] these are applied to delay-free systems with saturation. However, when delays occur in the control implementation, the closed-loop dynamics should be modelled as *retarded delay differential equations*, which, due to the non-smooth effect of saturation, will have a *non-smooth* right-hand side. While smooth, and in particular linear, retarded delay differential equations are relatively well-understood, cf. [7–9], few results are applicable to non-smooth retarded delay equations, and the nonsmooth nature of these equations necessitates more versatile analysis tools.

In [10,11], saturation is analysed using a polytopic overapproximation based on the observation that, given $H > 1$, the scalar

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saturation function $\text{sat}(y) = \text{sign}(y)\min(|y|, 1)$ satisfies $\text{sat}(y) \in [H^{-1}y, y]$ when $|y| < H$. Consequently, in the domain where $|y| < H$, the right-hand side of the non-smooth retarded delay differential equation can be over-approximated by a set of linear functions. Hence, the stability and convergence properties of the delay system are guaranteed with a linear retarded differential inclusion, whose stability properties are given by the generating vertices. Generalisations of this approach are given e.g. in [11]. Focusing on linear time-delay systems controlled by saturating *non-delayed* actuators, polytopic overapproximations of the functions $\text{sat}(y)$ or $y - \text{sat}(y)$ have been used as well in [12–15], leading to controller synthesis and H_∞ performance results, where both time-varying delays and neutral systems can be considered. Quadratic Lyapunov–Krasovskii functionals are used, such that only ellipsoidal basin of attraction estimates have been attained in these references.

We follow a different approach, and do not make a polytopic overapproximation of the saturation function. Instead, we exploit the observation that the saturation function induces a piecewise affine nature of the retarded differential equation, and analyse this dynamics with a piecewise quadratic Lyapunov–Krasovskii functional. For this purpose, firstly, we analyse the delay-free system with a piecewise quadratic Lyapunov function which is appropriate to identify the basin of attraction of the delay-free system. Addition of a functional term allows to study the basin of attraction of the closed-loop system with Lyapunov–Krasovskii techniques. An overapproximation of the difference $\text{sat}(Kx(t-\tau)) - \text{sat}(Kx(t))$ is used to evaluate the Lyapunov–Krasovskii functional along solutions.

The main contribution of this paper is the design of a piecewise quadratic Lyapunov–Krasovskii functional to find a basin of attraction estimate for saturated systems with constant input delay and input saturation. Combining the piecewise-quadratic nature of this functional with the piecewise-affine nature of the closed-loop dynamics allows to formulate delay-dependent conditions for the convergence of trajectories to the origin. This leads to two alternative computationally tractable conditions to find basin of attraction estimates. The first approach exploits an exponential stability bound for the delay-free case obtained using a quadratically constrained quadratic problem (QCQP), while the second approach allows to design the parameters of the Lyapunov–Krasovskii functional with a single line-search combined with the solution of a Linear Matrix Inequality. We discuss the interpretation of the attained basin of attraction, that is a set of initial functions, in this class of systems.

As our approach naturally leads to a conservative estimate of the basin of attraction, we also present simulations of two examples to assess the conservatism of the presented sufficient conditions and to compare with an existing method that uses a quadratic Lyapunov–Krasovskii functional.

The outline of the remainder of this paper is as follows. In the following section, we present the dynamical model and necessary notation. In Section 3, the basin of attraction is estimated for the delay-free system, and in Section 4, the employed Lyapunov function is used to analyse the delayed system and provide basin of attraction estimates. An example is presented in Section 5, and conclusions are given in Section 6.

2. Modelling and notation

Consider the class of linear systems with saturating actuators modelled as

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $u \in \mathbb{R}^m$ the inputs of the actuators that experience saturation. The inputs u experience a delay τ and are given by $u(t) = \text{sat}(Kx(t-\tau))$, $K \in \mathbb{R}^{m\times n}$ and $\text{sat}(y)_i := \text{sign}(y_i)\min(|y_i|, 1)$, $i = 1, \ldots, m$ for $y \in \mathbb{R}^m$. Hence, the closed-loop system is given by the nonsmooth retarded differential equation:

$$\dot{x}(t) = Ax(t) + B\text{sat}(Kx(t-\tau)). \tag{2}$$

Since (2) is a retarded differential equation, solutions should be considered in the state space of absolutely continuous functions. To describe these functions, we introduce the family of functions, parameterised with t, denoted by $x_\tau(t) : [-\tau, 0] \to \mathbb{R}^n$, such that $x_\tau(t)$ denotes the history of $x(t)$ in the time interval $[t-\tau, t]$. Hence, if $\varphi = x_\tau(t)$, then $\varphi(s) = x(t+s)$, $s \in [-\tau, 0]$. Let $AC([-\tau, 0], \mathbb{R}^n)$ denote the set of absolutely continuous mappings from $[-\tau, 0]$ to $\mathbb{R}^n$. Given a function $\nu : \mathbb{R}^n \to [0, \infty)$ and set $S \subset [0, \infty)$, let $\nu^{-1}(S)$ denote $\{x \in \mathbb{R}^n | \ \nu(x) \in S\}$. Let $\mathbb{R}_{\geq 0}$ denote the set of nonnegative scalars. Let O_{nm} denote an $n \times m$-dimensional matrix with zero elements, and let I_n denote an $n\times n$-dimensional identity matrix. Given $P \in \mathbb{R}^{n\times n}$, let $\text{He}(P)$ denote $P + P^T$ and $P \succ 0$ that P is symmetric and positive definite. $\|x\|_P^2$, with $x \in \mathbb{R}^n$, denotes x^TPx, $\|x\|$ is the Euclidean norm of x, x_i, $i = 1, \ldots, n$ the ith element of x and $\text{diag}(x)$ a diagonal matrix whose i, ith element equals x_i for $i = 1, \ldots, n$. We write $x > 0$ when $x_i > 0$ for all $i \in \mathbb{R}^n$. For a set $S \subset \mathbb{R}^n$, $\bar{S}$ denotes the closure of S, $\text{int}(S)$ the interior, ∂S its boundary and $\text{co}(S)$ the closed convex hull containing S.

3. Estimating the basin of attraction for the delay-free system

We first study the non-delayed system given by

$$\dot{x} = Ax + B\text{sat}(Kx). \tag{3}$$

Let the hypersurfaces $\{x|K_ix = \pm 1\}$, $i = 1, \ldots, m$ be used to introduce a polytopic partitioning $\{X_j\}$, $j \in \{1, \ldots, 3^m\}$ of $\mathbb{R}^n$ where, for $i = 1, \ldots, m$, K_i denotes the ith row of K and the sets X_j are closed. We use this partitioning to exploit the piecewise affine nature of (3).

To do so, for each polytope $X_j, j \in \{1, \ldots, 3^m\}$ we introduce the m-dimensional vector k^j such that the ith element of the vector k^j, i.e. k_i^j, equals -1 if $K_ix \leq -1$, $k_i^j = 0$ for $|K_ix| \leq 1$ and $k_i^j = 1$ for $K_ix \geq 1$ for $x \in X_j$. We observe that for $i \in \{1, \ldots, m\}$,

$$\text{sat}(Kx)_i = (1 - (k_i^j)^2)K_ix + k_i^j, \quad x \in X_j,\ j \in \{1, \ldots, 3^m\}, \tag{4}$$

such that (3) can be rewritten as

$$\dot{x} = \bar{A}_j\bar{x}, \quad x \in X_j,\ j = 1, \ldots, 3^m, \tag{5}$$

with $\bar{x} := \begin{pmatrix} x \\ 1 \end{pmatrix}$ and

$$\bar{A}_j := \begin{pmatrix} A + B(I - \text{diag}(k^j)^2)K & Bk^j \end{pmatrix}. \tag{6}$$

3.1. Design of a piecewise quadratic Lyapunov function

As the vector field (3) is piecewise affine, we propose a piecewise polynomial Lyapunov function with the same partitioning:

$$V_{nd}(x) = \bar{x}^T\bar{P}_j\bar{x}, \quad x \in X_j,\ j = 1, \ldots, 3^m, \tag{7}$$

cf. [4]. Here, the matrices $\bar{P}_j$ are related via

$$\bar{P}_j := \begin{pmatrix} \text{diag}^2(k^j)K & -k^j \\ I_n & O_{n1} \end{pmatrix}^T T \begin{pmatrix} \text{diag}^2(k^j)K & -k^j \\ I_n & O_{n1} \end{pmatrix}, \tag{8}$$

with a symmetric matrix $T \in \mathbb{R}^{(m+n)\times(m+n)}$. Note that the function V_{nd} is continuous as $\begin{pmatrix} \text{diag}^2(k^j)K & -k^j \end{pmatrix}\bar{x} = Kx - \text{sat}(Kx)$ for $x \in X_j$, cf. (4). In the polytope X_{j_0}, $j_0 \in \{1, \ldots, 3^m\}$, that contains the origin, we observe that $k^{j_0} = 0$. Hence, locally near the origin, the Lyapunov function is quadratic and given by $V_{nd}(x) = x^TP_0x$, with

$$P_0 := \begin{pmatrix} O_{nm} & I_n \end{pmatrix} T \begin{pmatrix} O_{nm} & I_n \end{pmatrix}^T. \tag{9}$$

3.2. Estimate of the delay-free basin of attraction

Given a positive definite Lyapunov function V_{nd} of the form (7), i.e. for fixed T, we can now provide a basin of attraction estimate for the delay free system by finding a sublevel set of V_{nd} where $\frac{dV_{nd}(x(t))}{dt} \le -\epsilon V_{nd}(x(t))$, with $\epsilon \ge 0$, for all trajectories $x(t)$ of (3), provided the sublevel set is a bounded set.

By application of Lemma 4 in Appendix A or the results in [16], we observe that $\frac{dV_{nd}(x(t))}{dt} = \bar{x}(t)^T \mathrm{He}\left(\bar{P}_j^1 \bar{A}_j\right)\bar{x}(t)$ for some $j \in \{1, \dots, 3^m\}$ such that $x(t) \in X_j$ and almost all t, with

$$\bar{P}_j^1 := \bar{P}_j \begin{pmatrix} I_n & O_{n1} \end{pmatrix}^T. \tag{10}$$

Hence, the following optimisation problem is attained:

$$\gamma_\epsilon := \min_{j\in\{1,\dots,3^m\}} \sup_{\gamma} \left\{ \gamma \,\middle|\, \begin{array}{l} V_{nd}(x) < \gamma \\ x \in X_j \end{array} \right. \\ \left. \Rightarrow \bar{x}^T \mathrm{He}\left(\bar{P}_j^1 \bar{A}_j\right)\bar{x} \le -\epsilon V_{nd}(x) \right\}. \tag{11}$$

The sublevel set

$$\{x \in \mathbb{R}^n |\, V_{nd}(x) < \gamma_\epsilon\} \tag{12}$$

contains initial conditions of trajectories that are attracted exponentially to the origin. We observe that the optimisation problem (11) is equivalent to

$$\gamma_\epsilon = \min_{j\in\{1,\dots,3^m\}} \inf_x \{\bar{x}^T \bar{P}_j \bar{x} |\, x \in X_j,\ \bar{x}^T \mathrm{He}(\bar{P}_j^1 \bar{A}_j)\bar{x} \ge -\epsilon \bar{x}^T \bar{P}_j \bar{x}\} \tag{13}$$

where (7) has been substituted and, as X_j are polytopes, the requirement $x \in X_j$ provides linear constraints on x. Therefore, the inner problem in (13), i.e., for a given j, is a quadratically constraint quadratic problem (QCQP).

Various methods exist to solve this (non-convex) QCQP, e.g. by exploiting the Karush–Kuhn–Tucker conditions for optimality, cf. in [17] for $m = 1$. Assuming V_{nd} is fixed a priori by choosing the matrix T, the scalar γ_ϵ in (11) will be used in Section 4.1 to estimate the basin of attraction for the delay system (2). Subsequently, we present an algorithm to design the function V_{nd} and derive the basin of attraction estimate in Section 4.2.

Alternatively, in [4], for a general class of piecewise affine delay-free systems, and in [5], for delay-free systems with a single saturating actuator, the $\mathcal{S}$-procedure is used such that the non-global condition $\bar{x}^T \mathrm{He}(\bar{P}_j^1 \bar{A}_j)\bar{x} \le -\epsilon V_{nd}(x),\ \forall x \in X_j \cap \{x | V_{nd}(x) < \gamma_\epsilon\}$, $j = 1, \dots, 3^m$, is replaced by a global condition $\bar{x}^T \mathrm{He}(\bar{P}_j^1 \bar{A}_j)\bar{x} + \bar{S}_j(\gamma_\epsilon, x) \le -\epsilon V_{nd}(x)$, $\forall x$, where each function $\bar{S}_j(\gamma_\epsilon, x)$ is a function that is positive for all γ_ϵ and all $x \in X_j$ such that $V_{nd}(x) \le \gamma_\epsilon$. Then,

$$\max \left\{ \bar{\gamma} \,|\, \bar{x}^T \mathrm{He}(\bar{P}_j^1 \bar{A}_j)\bar{x} + \bar{S}_j(\bar{\gamma}, x) \right. \\ \left. \le -\epsilon V_{nd}(x),\ \forall x, j = 1, \dots, 3^m \right\} \le \gamma_\epsilon \tag{14}$$

provides a lower bound for γ_ϵ, which, for the designs of $\bar{S}$ proposed in [5,4], can be computed using Linear Matrix Inequalities (LMIs). To minimise the difference between both sides of (14), the design of the function $\bar{S}$ has to be optimised, cf. [5,4]. In Section 4.2, we extend this approach to the system (2) with delay. The advantage of this approach is that the matrix T is a decision variable in the optimisation problem, such that the design of T, and therewith of the function V_{nd}, is optimised.

Remark 1. Similar piecewise quadratic Lyapunov functions also appeared in [6,18] to study delay-free saturating control systems. In [18], global stability is investigated, and in both papers, matrix T is required to be positive definite. This requirement is sufficient, but not necessary for V_{nd} to be positive definite. In [6], the property $T \succ 0$ is exploited to estimate basins of attraction in a single bilinear matrix inequality, therewith avoiding the iterative procedure of [4,5] where positive definiteness has to be checked in every polytope X_j. As we will illustrate in Example 2, however, allowing a non-positive definite matrix T can lead to better estimates of the basin of attraction.

4. Delay-dependent basin of attraction estimate

In this section, we analyse system (2) that encompasses delay using the results of the previous section. Namely, we introduce the Lyapunov–Krasovskii functional:

$$V(x_\tau(t)) = V_{nd}(x(t)) + W(x_\tau(t)), \tag{15a}$$

with V_{nd} in (7) and a nonnegative functional

$$W(x_\tau(t)) := \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}(\bar{s})^T K^T R K \dot{x}(\bar{s}) d\bar{s} d\theta, \tag{15b}$$

with $R \succ 0$.

In the following lemma, given an analysis domain D_a, we present sufficient conditions for

$$\mathcal{B}_{oa} := \{x_\tau \in AC([-\tau, 0], \mathbb{R}^n) |\, V(x_\tau) \le \Gamma,\ x_\tau(0) \in D_a\}, \tag{16}$$

with $\Gamma > 0$, to be contained in the basin of attraction. Here, the analysis domain is typically selected as $D_a = \mathbb{R}^n$ when V_{nd} is positive definite, otherwise, a compact set D_a containing the origin is selected such that $V_{nd}(x) > 0$ holds for all $x \in D_a \backslash \{0\}$. Note that the set $\mathcal{B}_{oa}$ is a subset in the space of initial functions. In the typical case where the rank of K is smaller than n, an infinite-dimensional and unbounded approximation of the basin of attraction is attained. In Section 4.4, we will discuss how this estimate can be related to a finite-dimensional set of initial conditions x_0, assuming that the control action is zero for $t < \tau$.

Recall that given $r \subset \mathbb{R}_{\ge 0}$, $V_{nd}^{-1}(r)$ denotes $\{x \in \mathbb{R}^n |\, V_{nd}(x) \in r\}$. By adapting the approach of [5] to the delay case considered here, we impose conditions guaranteeing decay of the Lyapunov–Krasovskii functional (i.e., (17)) for points x which are contained in $V_{nd}([0, \Gamma]) \cap D_a$, with $\Gamma > 0$ as large as possible and D_a an analysis domain that has to be optimised.

Lemma 1. *Let V be of the form (15), with V_{nd} in (7) and a symmetric matrix $T \in \mathbb{R}^{(n+1)\times(n+1)}$. Let V_{nd} be positive definite in a domain D_a containing the origin, with D_a either compact or $\mathbb{R}^n$, and $V_{nd}(x) > \Gamma$ for $x \in \partial D_a$, with $\Gamma > 0$. Let S_k, $k = 1, \dots, 2^m$ give the diagonal $m \times m$ matrices with zero or one at each diagonal element. If there exist matrices $R \in \mathbb{R}^{m\times m}$, $R \succ 0$ and $P^\circ \in \mathbb{R}^{n\times n}$ such that for all $j \in \{1, \dots, 3^m\}$, $k \in \{1, \dots, 2^m\}$,*

$$z^T \Xi_{jk} z < 0, \tag{17}$$

holds for all $z = \begin{pmatrix} z_x^T & 1 & z_d^T \end{pmatrix}^T$, where $z_x \in X_j \cap V_{nd}^{-1}([0, \Gamma]) \cap D_a \setminus \{0\}$, and $z_d \in \mathbb{R}^n$, with

$$\Xi_{jk} = \begin{pmatrix} He(\bar{P}_j^1 \bar{A}_j) & \bar{A}_j^T P^{\circ T} \\ P^\circ \bar{A}_j & He(P^\circ) + \tau K^T R K \end{pmatrix} \\ + \tau \begin{pmatrix} \bar{P}_j^1 B S_k \\ P^\circ B S_k \end{pmatrix} R^{-1} \begin{pmatrix} S_k B^T \bar{P}_j^{1T} & S_k B^T P^{\circ T} \end{pmatrix},$$

then all trajectories with initial conditions $x_\tau \in \{x_\tau |\, V(x_\tau) \le \Gamma\}$ converge to the origin.

Proof. See Appendix B.

In the next section, we will exploit this nonlinear semi-infinite inequality to attain computationally tractable procedures to estimate the basins of attraction, where the second term of Ξ_{jk} is resolved using Schur's complement.

Remark 2. The result in Lemma 1 is obtained by rewriting (2) as $\dot{x}(t) = \bar{A}_i\bar{x}(t) + B(\mathrm{sat}(Kx(t-\tau)) - \mathrm{sat}(Kx(t)))$ for $x(t) \in X_i$, and overapproximating the second term. Effective convex-hull representations of the saturation function have been presented e.g. in [19,20]. In contrast to these works, in Lemma 1, we overapproximate the difference between two saturated functions and find

$$\mathrm{sat}(Kx(t-\tau)) - \mathrm{sat}(Kx(t)) \in \mathrm{co}\{S_kK(x(t-\tau) - x(t)),\ k = 1, \dots, 2^m\}. \tag{18}$$

Namely, the single-valued saturation function satisfies a Lipschitz constant 1 and is non-decreasing, such that $\mathrm{sat}(y_1) - \mathrm{sat}(y_2) \in \mathrm{co}\{0, y_1 - y_2\} = \{s(y_1 - y_2)|\ s \in \{0, 1\}\}$ if $y_1, y_2 \in \mathbb{R}$. Hence, for $Y_1, Y_2 \in \mathbb{R}^m$, we find $\mathrm{sat}(Y_1) - \mathrm{sat}(Y_2) \in \mathrm{co}\{S_k(Y_1 - Y_2),\ k = 1, \dots, 2^m\}$. In the limit of $\tau \to 0$, the conservatism introduced in (18) vanishes as $\lim_{t\to 0}(x(t-\tau) - x(t)) = 0$.

Remark 3. As K^TRK is typically not positive definite, Lemma 1 guarantees convergence to the origin with a Lyapunov–Krasovskii functional V that is not positive definite for all $x_\tau \in \{y \in AC([-\tau, 0], \mathbb{R}^n)|\ y(0) \in D_a\}$. This can be understood from the observation that $AC([-\tau, 0], \mathbb{R}^n)$ is not a minimal state space, as solutions can be continued uniquely from time t when $x(t)$ and the function $u(s) = Kx(s),\ s \in [t-\tau, t]$, are known.

In the literature, see e.g. [21], the functional $W(x_\tau(t)) := \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}(\bar{s})^T\bar{R}\dot{x}(\bar{s})d\bar{s}d\theta$, has been used, with a positive definite matrix $\bar{R} \in \mathbb{R}^{n\times n}$. The observation that the functional should provide a bound on $K\dot{x}(s),\ s \in [-\tau, 0]$ motivates the selection $\bar{R} = K^TRK$ used in this paper.

We now present two computationally tractable conditions that guarantee the conditions of Lemma 1. First, in Section 4.1, the inequality $\bar{x}^T\mathrm{He}(\bar{P}_j^1\bar{A}_j)\bar{x} \le -\epsilon V_{nd}(x),\ x \in X_j,\ j = 1, \dots, 3^m$ in (11) will be used to estimate the basin of attraction for the delay system (2), assuming V_{nd} is fixed a priori. Second, in Section 4.2, we use the $\mathcal{S}$-procedure to find sufficient conditions for (17) and synthesise the function V_{nd}, restricting our attention a priori to an ellipsoidal analysis domain D_a.

4.1. Basin of attraction estimate using delay-free result

The results of Section 3, and, in particular, (11), allow to compute a basin of attraction estimate when V_{nd} is given. More precisely, it allows to replace $\mathrm{He}(\bar{P}_j^1\bar{A}_j)$ in (17) with $-\epsilon\bar{P}_j$ and use $D_a = \mathbb{R}^n$, yielding the following result.

Theorem 2. *Let T be such that V_{nd} in (7) is positive definite, let $\epsilon > 0$ and $\Gamma = \gamma_\epsilon$ be obtained by solving (11). Let $\{S_k\}_{k=1,\dots,2^m}$, denote the set of diagonal $m \times m$ matrices with zero or one at each diagonal element. If there exist matrices $R \in \mathbb{R}^{m\times m}$, $P^\circ \in \mathbb{R}^{n\times n}$, with $R \succ 0$, and a scalar $\delta_1 > 0$ such that*

$$\begin{pmatrix} (\delta_1 - \epsilon)\bar{P}_j & \bar{A}_j^T P^{\circ T} & \sqrt{\tau}\bar{P}_j^1BS_k \\ P^\circ\bar{A}_j & -He(P^\circ) + \tau K^TRK & \sqrt{\tau}P^\circ BS_k \\ \sqrt{\tau}S_kB^T\bar{P}_j^{1T} & \sqrt{\tau}S_kB^TP^{\circ T} & -R \end{pmatrix} \preceq 0, \tag{19}$$

holds for all $j \in \{1, \dots, 3^m\}$, $k \in \{1, \dots, 2^m\}$, then all trajectories with initial conditions in $\{x_\tau|\ V(x_\tau) \le \Gamma\}$ converge to the origin, where V is given in (15).

Proof. See Appendix B.

The parameter δ_1 can be chosen arbitrary small, e.g. one order larger than the numerical accuracy, such that it does not have to be selected by a numerical solver. The introduction of this parameter cannot be avoided by making the matrix inequality strict. Namely, typically, the matrix $\bar{P}_j$ is not positive definite for every j, such that a strict matrix inequality (19) is not feasible.

Given a Lyapunov function for the delay-free system, this theorem allows a computationally tractable procedure to find a basin of attraction estimate for system (2). Namely, using a line-search in ϵ, one can find the minimum ϵ such that the linear matrix inequalities in the theorem are satisfied. Subsequently, the basin of attraction given in (16) is used, with $\Gamma = \gamma_\epsilon$ and γ_ϵ in (11).

4.2. Basin of attraction estimate with a pre-defined analysis domain

As an alternative approach, the $\mathcal{S}$-procedure can be employed to find computationally tractable conditions that guarantee the conditions in Lemma 1. In what follows, the matrix T does not need to be fixed beforehand. In fact, we will present a procedure to construct a matrix T that may not be positive definite and, furthermore, may imply that the function V_{nd} can be negative outside the analysis domain D_a.

Condition (17) is guaranteed by adding functions $\bar{S}_j(x(t)),\ j = 1, \dots, 3^m$, that are positive for $x \in X_j \cap D_a$, to (17), and proving that the sum is negative definite for all z. Similar to the approach of [5], we design functions $\bar{S}_j$ using the ellipsoidal analysis domain $D_a = \{x|\ x^TP_ax \le t\}$, where $t > 0$, $P_a \succ 0$, and select $\bar{S}_j(x) = \bar{E}_j^TW_j\bar{E}_j + w_j\begin{pmatrix} -P_a & 0_{n1} \\ 0_{1n} & t \end{pmatrix}$, with $w_j > 0$ and W_j symmetric matrices with nonnegative elements. The matrices $\bar{E}_j \in \mathbb{R}^{s_j\times(n+1)}$ $j = 1, \dots, 3^m$ are such that $\bar{E}_j\bar{x} \ge 0$ for all $x \in X_j \cap D_a$. These matrices can be computed as in [5]. If $\frac{dV(x_\tau(t))}{dt}$ is negative in the domain D_a, then a basin of attraction estimate as in (16) is found, where Γ is the maximum number such that $V_{nd}(y) > \Gamma$ for all $y \in \partial D_a$. In this manner, the following theorem is obtained.

Theorem 3. *Let $\{S_k\}_{k=1,\dots,2^m}$ denote the set of diagonal $m \times m$ matrices with zero or one at each diagonal element. Let an analysis domain $D_a = \{x|\ x^TP_ax \le t\}$, $t > 0$, $P_a \succ 0$ be given and let the matrices $\bar{E}_j \in \mathbb{R}^{s_j\times(n+1)}$, $j = 1, \dots, 3^m$ be such that $\bar{E}_j\bar{x} \ge 0$ for all $x \in X_j \cap D_a$. Let the scalar $\delta_1 > 0$.*

If there exist symmetric matrices $T \in \mathbb{R}^{(m+n)\times(m+n)}$, $U_j, W_j \in \mathbb{R}^{s_j\times s_j}$, where U_j, W_j have nonnegative elements, scalars $u_j, w_j > 0$, matrices $P^\circ \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, $R \succ 0$ such that for all $j \in \{1, \dots, 3^m\}$, $k \in \{1, \dots, 2^m\}$,

$$\bar{P}_j - \bar{E}_j^T\bar{U}_j\bar{E}_j - u_j\begin{pmatrix} -P_a & 0_{n1} \\ 0_{1n} & t \end{pmatrix} \succ 0 \tag{20a}$$

$$\begin{pmatrix} \Psi_j & \bar{A}_j^TP^{\circ T} & \sqrt{\tau}\bar{P}_j^1BS_k \\ P^\circ\bar{A}_j & -He(P^\circ) + \tau K^TRK & \sqrt{\tau}P^\circ BS_k \\ \sqrt{\tau}S_kB^T\bar{P}_j^{1T} & \sqrt{\tau}S_kB^TP^{\circ T} & -R \end{pmatrix} \preceq 0, \tag{20b}$$

hold, with $\Psi_j = He(\bar{P}_j^1\bar{A}_j) + \delta_1\bar{P}_j + \bar{E}_j^TW_j\bar{E}_j + w_j\begin{pmatrix} -P_a & 0_{n1} \\ 0_{1n} & t \end{pmatrix}$ and $\bar{P}_j, \bar{P}_j^1$ depending linearly on T as given in (10), (8), then all trajectories from $\mathcal{B}_{oa}$ converge to the origin, with Γ selected such that $V_{nd}(y) > \Gamma,\ \forall y \in \partial D_a$.

Proof. See Appendix B.

Note that this theorem is also applicable when V_{nd} in (7) is not positive definite, cf. Example 2. This theorem leads to the following procedure to find a basin of attraction estimate.

Algorithm 1.

1. Find a positive definite matrix P_a using Proposition 1 of [5], which is based on the delay-free case.
2. Fix δ_1 and perform a line search to find the maximal t satisfying the conditions of Theorem 3. For each iteration,
   (a) compute the matrices $\bar{E}_j$
   (b) search for matrices $R, \bar{U}_j, W_j, T, P^\circ$ and scalars u_j, w_j, with $j = 1, \dots, 3^m$, for which (20) holds.

3. Find the maximal Γ such that $V_{nd}(y) > \Gamma$ for $y \in \partial D_a$ with the LMI given in [5, Proposition 2].

Remark 4. Theorem 3 can be used to *design* the piecewise quadratic function V_{nd}, as the matrix conditions are linear in T, R, u_j, U_j, w_j, W_j and P°, and δ_1 can be fixed a priori by a small number. Following [5], in Section 5, basin of attraction estimates are attained by minimising $\sum_{j=0}^{3^m} \mathrm{tr}(P_j) + \tau R$, with $P_j = \begin{pmatrix} I_n & O_{n1} \end{pmatrix} \bar{P}_j \begin{pmatrix} I_n & O_{n1} \end{pmatrix}^T$.

Remark 5. The analysis domain has to be fixed a priori and cannot be found with LMI optimisation techniques. An iterative procedure to find a suitable domain is given in [5] for the delay-free case. Alternatively, one may require $T \succ 0$, which is the approach used in [6]. Then, the analysis domain can be designed with a combined optimisation problem, expressed as an LMI, where both the analysis domain and R, T, P° as given in Lemma 1 are defined.

4.3. Comparison between both approaches

Comparing Theorems 2 and 3, we observe that conservatism is introduced in both results via different effects. In Theorem 2, the overapproximation $\nabla V_{nd}(x(t))\bar{A}_i\bar{x}(t) \le -\epsilon V_{nd}(x)$, $x \in X_i$, $i = 1, \ldots, 3^m$, introduces conservatism, whereas in Theorem 3, the Lyapunov–Krasovskii functional is required to decrease for all $x(t)$ in an analysis domain D_a, which is not necessary as decrease is only needed when $x(t) \in V_{nd}^{-1}([0, \Gamma]) \cap D_a$. In both approaches, conservatism is introduced as well by the particular Lyapunov–Krasovskii functional given in (15).

In contrast to Theorem 2, Theorem 3 allows to design the parameter T that determines V_{nd}. However, given a design for the Lyapunov function V_{nd} for the delay-free case, the conditions in Theorem 2 have a lower computational cost, since the $\mathcal{S}$-procedure in Theorem 3 introduces additional variables and constraints.

4.4. Basin of attraction estimate for control implementation

In various control applications, measurement and actuation of the control system (2) are started at the same time instant $t = 0$. However, during the time interval $t \in [0, \tau]$, no actuation can be applied due to the delay, such that $u(t) = 0$ is acting on the system and $x(t) = e^{At}x_0$, $t \in [0, \tau]$. Hence, the state $x_\tau(\tau)$ of system (2) depends on $x_0 \in \mathbb{R}^n$ only, and we can characterise the basin of attraction in terms of x_0 by exploiting the bounded growth in the time interval $t \in [0, \tau]$, [22]. In this manner, the following result can be attained from Theorem 3.

Corollary 1. *Consider the system (2) with $\tau > 0$. Let V_{nd} be given in (7), R as in Theorem 3 and $Z = \int_{-\tau}^{0}\int_{\tau+\theta}^{\tau} e^{A^T(\tau+s)}A^TK^TRKAe^{A(\tau+s)}dsd\theta$. If V_{nd}, D_a R, τ and Γ satisfy the conditions in Theorem 3, then trajectories of (2) with initial state x_0, $u(t) = 0$ for $t \in [0, \tau]$ are attracted to the origin when $x_0 \in \{x_0 \in \mathbb{R}^n | V_{nd}(e^{A\tau}x_0) + x_0^TZx_0 \le \Gamma\}$.*

Clearly, an analogue result can be obtained by replacing Theorem 3 with Theorem 2.

5. Numerical examples and comparison with existing methods

Example 1. Let $A = \begin{pmatrix} 0 & 1 \\ -0.2 & 0.05 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $K = \begin{pmatrix} -0.25 & -0.2 \end{pmatrix}$. By adopting Algorithm 1, that implements Theorem 3, for a range of delay values τ, the basin of attraction estimates in Fig. 1 have been obtained, where $P_a = \begin{pmatrix} 3.24 & 0.53 \\ 0.53 & 7.88 \end{pmatrix}$ and $\delta_1 = 10^{-6}$ is used. These basins of attraction are depicted for varying initial position x_0 and initial input signal $u(t) = 0$ for $t \in [0, \tau]$, as described in Corollary 1. To assess the conservatism of this approach, the unstable limit cycle that limits the basin of attraction in the delay-free case is also depicted. Note that for $\tau = 0.5$, the estimate attained with Corollary 1 is strictly contained within the saturation bounds. However, in the time interval from $t = 0$ until $t = \tau$, trajectories from this domain will contain trajectory segments where the control action is saturated. For $\tau = 0$, the Lyapunov function V_{nd} is given by (7) with $T = \begin{pmatrix} -1.1447 & 0.0015 & -0.0095 \\ 0.0015 & 0.1403 & 0.0426 \\ -0.0095 & 0.0426 & 0.3657 \end{pmatrix}$ and $\Gamma = 27.64$. Note that while $T \not\succ 0$, the function V_{nd} is positive definite.

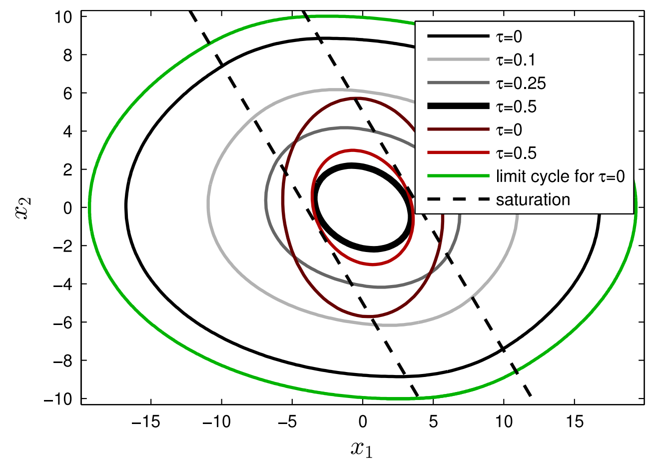

Fig. 1. Basin of attraction estimates for Example 1 from Algorithm 1 (in black and grey) and from the approach of [23] in shades of red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

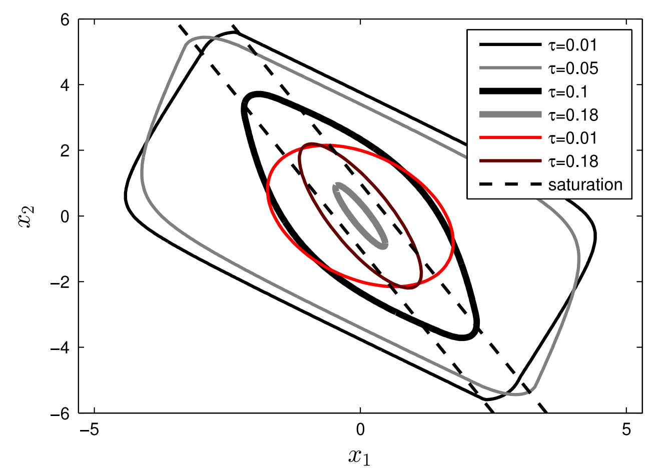

Fig. 2. Basin of attraction estimates for Example 2 found with Algorithm 1 in grey and black, and from the approach of [23] in shades of red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Example 2. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$ and $K = \begin{pmatrix} 2 & 1 \end{pmatrix}$, which, for $\tau = 0$, correspond to Example 1 in [5]. In Fig. 2, the basin of attraction estimates are shown for a range of delay values τ. Here, $P_a = \begin{pmatrix} 1.79 & 0.86 \\ 0.86 & 0.48 \end{pmatrix}$ and $\delta_1 = 10^{-6}$ is chosen in Algorithm 1 and Corollary 1 is applied, yielding a set of initial positions x_0 and initial input signals $u(t) = 0$ for $t \in [0, \tau]$, that are contained in the basin of attraction. To illustrate the conservatism of the attained basin of attraction estimates, we note that for all τ, the system has equilibria at the origin and at $x = \begin{pmatrix} \pm 5 & 0 \end{pmatrix}^T$. For $\tau = 0$, the stable manifolds of the latter equilibria form the boundary of the basin of attraction. For $\tau = 0.01$, $T = \begin{pmatrix} -2.4106 & 1.4111 & 0.8954 \\ 1.4111 & 3.3185 & 1.3217 \\ 0.8954 & 1.3217 & 0.9870 \end{pmatrix}$, $t = 44.81$ and $\Gamma = 14.58$ are found, such that T is not positive definite. Moreover, in this example, V_{nd} is not everywhere positive outside the analysis domain. For instance, $V_{nd}(\begin{pmatrix} 10 & 0 \end{pmatrix}^T) = -2.1529$.

Comparison with existing methods

The proposed method provides an estimate of the basin of attraction for systems with actuator saturation and actuator delay, while most results in the literature consider systems where the saturating signal is delay-free, such that the closed-loop dynamics is $\dot{x}(t) = Ax(t) + A_\tau(x(t-\tau)) + B\mathrm{sat}(Kx(t))$, cf. e.g. [24,13,15,25]. An exception is the recent paper [23], which presents an approach to find ellipsoidal basin of attraction estimates for (2) where, firstly, 5 optimisation parameters have to be fixed, and, subsequently, an LMI is solved with $(3\frac{n(n+1)}{2} + 4 + 4n^2 + nm)$ free variables. For comparison, in Step 1 of Algorithm 1, 1 prefixed variable and $\frac{n(n+1)}{2}$ free variables for the LMI problem are found. In step 2, $\frac{(n+m)(n+m+1)}{2} + \frac{m(m+1)}{2}$ free variables define the functional V, $(3^m - 1)((2m+1)(2m+2)+2)$ variables are introduced with the $\mathcal{S}$-procedure, and the number δ_1 has to be fixed. For Step 3, $\frac{3^m((2^m+1)(2^m)+2)}{2}$ free variables are used. Here, we followed the polytopic partitioning presented in [5].

The number of LMIs can be reduced by exploiting symmetry of the polytopic conditions (e.g. if $m = 1$, it suffices to check conditions in one central, and one outer polytope). Furthermore, we envision it may be possible to reduce the number of polytopes further by developing a piecewise quadratic Lyapunov–Krasovskii functional which is smooth in a lower number of polytopes generated only by hypersurface corresponding to the saturating inputs that most strongly affect the basin of attraction. At the cost of increased conservatism, we can replace the conditions, imposed in each polytope, that the Lyapunov–Krasovskii functional is decreasing and positive, with a single requirement and a regional sector condition, as proposed in [26] for the delay free case. Such approach may reduce the computational cost, while a smaller basin of attraction estimate is expected since the sector condition is expected to be more conservative.

We note that Algorithm 1 requires solving these LMIs in a line search, which was also required in the examples of [23] for one of the optimisation parameters.

In Fig. 1, the resulting basins of attractions are given in red for the method of [23], where, following the design approach of this reference, the optimisation parameters are selected as $(\psi_0, \psi_t, \omega_i, \omega_y, \epsilon_i) = (0.5, 0, 1, 1000, 15)$. For small values of the time-delay τ, the method proposed in the present paper attains a far larger basin of attraction estimate, which can be understood as the ellipsoidal estimates of [23] have less flexibility to approximate the true shape of the basin of attraction. In contrast, for larger time-delays, the method in [23] outperforms the presented algorithm, since more decision variables are involved to reduce the conservatism introduced by over-approximating the effect of the delay.

6. Conclusion

A method has been presented that provides an estimate of the basin of attraction of linear systems controlled by multiple saturating controllers with the same time-delay. A novel piecewise quadratic Lyapunov–Krasovskii functional is introduced which exploits the piecewise affine nature of the retarded delay differential equation that describes the closed-loop system.

Using this Lyapunov–Krasovskii functional, an estimate for the basin of attraction has been attained. To design this functional, we exploit the property that the space of absolutely continuous functions is not a minimal state space, as only the control action $Kx(t-\tau)$ of the system experiences delay. Hence, the basin of attraction estimate is an unbounded set of absolutely continuous functions. If the control action is zero before activation of the feedback, then the basin of attraction can be characterised as a bounded set of the initial states $x(0) \in \mathbb{R}^n$ only.

The conditions presented in this paper can effectively be employed to provide estimates of the basins of attraction, as has been illustrated for two examples.

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Appendix A. Time derivative of a piecewise smooth function along solutions of an ODE

The following technical result allows to evaluate a continuous Lyapunov function, that is differentiable inside polytopes, along the trajectories of a non-autonomous differential equation (see e.g. [27] for similar results for Lipschitz functions).

Lemma 4. *Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous and piecewise smooth function given by $V(x) = V_i(x)$ for $x \in X_i$, $i = 1, 2, \ldots$, with $V_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuously differentiable functions, polytopes X_i with non-intersecting interiors and $\bigcup_{i=1,2,\ldots} X_i = R^n$. For solutions $x(t)$ satisfying $\dot{x} = f(t,x)$ a.e., with f a function that is Lipschitz in x and measurable in t, it holds that $V(x(t))$ is absolutely continuous and the time derivative of this function is given by*

$$\frac{dV(x(t))}{dt} = \nabla V_{j(x(t))}(x(t))f(t,x(t)), \quad a.e., \tag{A.1}$$

with $j(x)$ a function that denotes, for every x, the minimum integer such that $V(x) = V_{j(x)}(x)$.

Proof. Since $x(t)$ is a solution to the differential equation $\dot{x} = f(t,x)$, the function $x : [0,\infty) \to \mathbb{R}^n$ is absolutely continuous. Hence, as V is a Lipschitz function, $V(x(t))$ is absolutely continuous and $\frac{dV(x(t))}{dt}$ is defined almost everywhere, i.e., for all $t \in \mathbb{R} \setminus \Omega$, with Ω a set of measure zero.

We now use proof by contradiction to show that the equality (A.1) holds for almost all $t \in \mathbb{R}$. Assume that there exists a connected time interval $I \subset \mathbb{R} \setminus \Omega$ with positive Lebesgue measure such that

$$\frac{dV(x(t))}{dt} \neq \nabla V_{j(x)}(x(t))f(t,x(t)), \quad \forall t \in I. \tag{A.2}$$

With $j(x)$ as in the lemma, we can partition the interval I in connected subintervals $\{I_k^c\}_{k=1,2,\ldots}$ such that in every subinterval I_k^c, the function $j(x(t))$ takes a constant value, that we denote with j^k. At least one of these subintervals, one of which we denote with $I_{\bar{k}}^c$, has a positive Lebesgue measure. For this time interval, we observe that $V(x(t)) = V_{j^{\bar{k}}}(x(t))$, and, as $V_{j^{\bar{k}}}$ is continuously differentiable and $x(t)$ satisfies $\dot{x}(t) = f(t,x(t))$ for all $t \in I_{\bar{k}}^c \subset I$, we attain $\frac{dV(x(t))}{dt} = \frac{dV_{j^{\bar{k}}}(x(t))}{dt} = \nabla V_{j(x)}(x(t))f(t,x(t))$, such that a contradiction with (A.2) is attained. Hence, (A.1) holds for almost all $t \in \mathbb{R}$. □

Appendix B. Proofs

Proof of Lemma 1. First, we observe that the function $\bar{V}_{nd}(x(t)) = \begin{cases} V_{nd}(x), & x \in D_a, \\ \infty & x \notin D_a \end{cases}$ is a positive definite and, when restricted to $\bar{V}_{nd}^{-1}([0, \Gamma]) \cap D_a$, continuous function.

As $\bar{V}_{nd}$ is positive definite and piecewise quadratic in D_a, we observe that there exists a $k_1 > 0$ such that

$$\bar{V}_{nd}(x) \geq k_1\|x\|^2, \quad \forall x \tag{B.1}$$

holds. Let $\bar{V}(x_\tau(t)) = \bar{V}_{nd}(x(t)) + W(x_\tau(t))$. As W is nonnegative, we observe that the $\bar{V}$ is nonnegative and $\bar{V}(x_\tau) \leq \Gamma$ implies $V_{nd}(x(0)) \leq \Gamma, x(0) \in D_a$. We will now evaluate $\bar{V}$ along a trajectory $x(t)$ of (2) with arbitrary initial conditions $x_\tau \in \mathcal{B}_{oa}$ and prove that $\bar{V}$ converges to zero along $x(t)$.

Direct evaluation of the time derivative of W along solutions $x_\tau(t)$ of (2) yields

$$\frac{dW(x_\tau(t))}{dt} = \tau\dot{x}(t)^T K^T RK\dot{x}(t) - \int_{t-\tau}^{t} \dot{x}(s)^T K^T RK\dot{x}(s)ds. \tag{B.2}$$

Following [21], we now present a nonnegative term to compensate the first term of (B.2). Observe with (2) that $x(t) \in \bar{X}_j$ implies that, for any $P^\circ \in \mathbb{R}^{n\times n}$,

$$0 = 2\dot{x}(t)^T P^\circ\big(-\dot{x}(t) + \bar{A}_j x(t) + B(\mathrm{sat}(Kx(t-\tau)) - \mathrm{sat}(Kx(t)))\big), \quad x(t) \in X_j. \tag{B.3}$$

To evaluate V_{nd} along $x(t)$, we first apply the methods of steps, cf. [7], and observe that, for any time interval $t \in [(\ell-1)\tau, \ell\tau]$, with ℓ an arbitrarily chosen integer, the solution $x(t)$ satisfies the time-dependent ordinary differential equation $\dot{x} = f(t,x) := \bar{A}_j\bar{x} - B\mathrm{sat}(Kx) + y_\ell(t)$, $x \in X_j, j = 1, \ldots, 3^m$, with $y_\ell(t) = B\mathrm{sat}(K(x(t-\tau)))$ a fixed time-varying function.

We now first evaluate the time derivative of $V_{nd}(x(t))$ when V_{nd} is differentiable, i.e. when $x(t) \in \mathrm{int}(X_j)$ for some j and $V_{nd}(x(t)) = V_j(x(t)) := \bar{x}(t)^T\bar{P}_j\bar{x}(t)$. Then, with $\nabla V_j(x(t)) = 2\bar{x}(t)^T\bar{P}_j^1$, we find

$$\nabla V_j f(t,x) = 2\bar{x}(t)^T\bar{P}_j^1\left(\bar{A}_j\bar{x}(t) + B\left(\mathrm{sat}(Kx(t-\tau)) - \mathrm{sat}(Kx(t))\right)\right). \tag{B.4}$$

As $\nabla V_j(x)$ and $f(t,x)$ are continuous in x, so is $\nabla V_j(x)f(t,x)$. Since, in addition, $\bar{X}_j \cap \bar{V}_{nd}^{-1}([0,\Gamma])$ is compact, cf. (B.1), we find that (B.4) holds for all $x \in \bar{X}_j \cap \bar{V}_{nd}^{-1}([0,\Gamma])$.

Applying Lemma 4 for the function V_{nd}, we observe that $V_{nd}(x(t))$ is absolutely continuous for all $t \in [(\ell-1)\tau, \ell\tau]$, and, in addition, (A.1) holds, with $j(x)$ defined in Lemma 4. As $j(x) \in \{\bar{j} \in \{1,\ldots,3^m\},\ x \in \bar{X}_{\bar{j}}\}$ for all $x \in V_{nd}^{-1}([0,\Gamma])$, combination of (A.1), and (B.4) yields that at each point $x(t) \in \bar{V}_{nd}^{-1}([0,\Gamma])$, there exists $j \in \{1,\ldots,3^m\}$ such that $x(t) \in \bar{X}_j$ and

$$\frac{dV_{nd}(x(t))}{dt} = 2\bar{x}(t)^T\bar{P}_j^1\left(\bar{A}_j\bar{x}(t) + B\left(\mathrm{sat}(Kx(t-\tau)) - \mathrm{sat}(Kx(t))\right)\right). \tag{B.5}$$

Given (15), summation of (B.2), (B.3) and (B.5) yields:

$$\frac{d\bar{V}(x_\tau(t))}{dt} = z(t)^T\begin{pmatrix}\mathrm{He}(\bar{P}_j^1\bar{A}_j) & \bar{A}_j^T P^{\circ T} \\ P^\circ\bar{A}_j & -\mathrm{He}(P^\circ) + \tau K^T RK\end{pmatrix}z(t) - \int_{t-\tau}^{t}\dot{x}(s)^T K^T RK\dot{x}(s)ds + 2z(t)^T\begin{pmatrix}\bar{P}_j^1 B \\ P^\circ B\end{pmatrix}(\mathrm{sat}(Kx(t-\tau)) - \mathrm{sat}(Kx(t))), \tag{B.6}$$

with $z(t) = \begin{pmatrix}\bar{x}(t)^T & \dot{x}(t)^T\end{pmatrix}^T$ and some j such that $x(t) \in X_j$. With (18), the last term can be overapproximated to obtain

$$\frac{d\bar{V}(x_\tau(t))}{dt} \leq z(t)^T\begin{pmatrix}\mathrm{He}(\bar{P}_j^1\bar{A}_j) & \bar{A}_j^T P^{\circ T} \\ P^\circ\bar{A}_j & -\mathrm{He}(P^\circ) + \tau K^T RK\end{pmatrix}z(t) - \int_{t-\tau}^{t}\dot{x}(s)^T K^T RK\dot{x}(s)ds + \int_{-\tau}^{0} -2z(t)^T\begin{pmatrix}\bar{P}_j^1 B \\ P^\circ B\end{pmatrix}S_k K\dot{x}(s)ds. \tag{B.7}$$

Overapproximating the integrand using $-2v^T u \leq u^T Ru + v^T R^{-1}v$, we find

$$\frac{d\bar{V}(x_\tau(t))}{dt} \leq z(t)^T\begin{pmatrix}\mathrm{He}(\bar{P}_j^1\bar{A}_j) & \bar{A}_j^T P^{\circ T} \\ P^\circ\bar{A}_j & -\mathrm{He}(P^\circ) + \tau K^T RK\end{pmatrix}z(t) + \tau z(t)^T\begin{pmatrix}\bar{P}_j^1 BS_k \\ P^\circ BS_k\end{pmatrix}R^{-1}\begin{pmatrix}S_k B^T\bar{P}_j^{1T} & S_k B^T P^{\circ T}\end{pmatrix}z(t). \tag{B.8}$$

Hence, condition (17) proves $\frac{d\bar{V}(x_\tau(t))}{dt} < 0$. Consequently, $\bar{V}$ is decreasing along trajectories $x_\tau(t)$ and $\bar{V}(x_\tau(0)) \leq \Gamma$ implies $\bar{V}(x_\tau(t)) \leq \Gamma, \forall t \geq 0$. Hence, for all trajectories of (2) with initial conditions $x_\tau \in \mathcal{B}_{oa}$, we find that firstly, $x(t) \in \bar{V}_{nd}^{-1}([0,\Gamma])$ for all $t \geq 0$, and secondly, $\bar{V}(x_\tau(t))$ converges monotonically to zero: for every $\epsilon_V > 0$, one can find a $T > 0$ such that $\bar{V}(x_\tau(t)) < \epsilon_V$ for all $t \geq T$. Since W is non-negative, this also implies that $\bar{V}_{nd}(x(t)) < \epsilon_V$ for $t \geq T$. With (B.1), we find $\|x(t)\|^2 \leq \frac{\epsilon_V}{k_1}$ for all $t \geq T$, which implies that $\|x_\tau(t)\|_\tau := \sup_{s\in[-\tau,0]}\|x(t+s)\| \leq \frac{\epsilon_V}{k_1}$ for all $t \geq T+\tau$. As ϵ_V can be chosen arbitrarily small, convergence of x_τ to zero is proven. □

Proof of Theorem 2. We prove the theorem by showing that Lemma 1 can be applied with $D_a = \mathbb{R}^n$.

Taking the Schur complement from (19), we obtain

$$\begin{pmatrix}(\delta_1-\epsilon)\bar{P}_j & \bar{A}_j^T P^{\circ T} \\ P^\circ\bar{A}_j & -\mathrm{He}(P^\circ) + \tau K^T RK\end{pmatrix} \tag{B.9}$$

$$+ \tau\begin{pmatrix}\bar{P}_j^1 BS_k \\ P^\circ BS_k\end{pmatrix}R^{-1}\begin{pmatrix}S_k B^T\bar{P}_j^{1T} & S_k B^T P^{\circ T}\end{pmatrix} \preceq 0. \tag{B.10}$$

From the definition of γ_ϵ in (11), we conclude $\bar{x}(t)^T\mathrm{He}(P_j A_j)\bar{x}(t) \leq -\epsilon\bar{x}(t)^T P_j\bar{x}(t)$ for $x(t) \in V_{nd}^{-1}([0,\gamma_\epsilon]) \cap X_j$. Hence, we can conclude $z(t)^T\Xi_{jk}z(t) \leq -\delta_1 V_{nd}(x(t)) < 0$ for $z(t)$, Ξ_{jk} as given in Lemma 1. Consequently, (17) can be applied and Lemma 1 is satisfied, proving the theorem. □

Proof of Theorem 3. We prove the theorem by application of Lemma 1. Taking the Schur complement of (20b), we find

$$\begin{pmatrix}\Psi_j & \bar{A}_j^T P^{\circ T} \\ P^\circ\bar{A}_j & -\mathrm{He}(P^\circ) + \tau K^T RK\end{pmatrix} + \tau\begin{pmatrix}\bar{P}_j^1 BS_k \\ P^\circ BS_k\end{pmatrix}R^{-1}\begin{pmatrix}S_k B^T\bar{P}_j^{1T} & S_k B^T P^{\circ T}\end{pmatrix} \preceq 0. \tag{B.11}$$

Since, in addition, $\bar{x}^T\bar{E}_j^T W_j\bar{E}_j\bar{x} \geq 0$ and $\bar{x}^T\begin{pmatrix}-P_a & 0_{n1} \\ 0_{1n} & t\end{pmatrix}\bar{x} \geq 0$for all $x \in D_a \cap V_{nd}^{-1}([0,\Gamma]) \cap X_j$, (20b) implies (17). Furthermore, (20a) implies that $V_{nd}(x)$ is positive definite for $x \in D_a$. Hence, Lemma 1 can be applied, proving the theorem. □

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